CONNECTING THE PROFINITE COMPLETION AND THE CANONICAL EXTENSION USING DUALITY

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ABSTRACT. We show using duality and category theory that the profinite completion $\hat{\mathbb{A}}$ of a bounded distributive lattice expansion $\mathbb{A}$ is a homomorphic image of the canonical extension $\mathbb{A}^\sigma$. Moreover the natural mapping $\mu\colon \mathbb{A} \to \hat{\mathbb{A}}$ can be extended to a surjection $\nu\colon \mathbb{A}^\sigma \twoheadrightarrow \hat{\mathbb{A}}$.

1. INTRODUCTION

The profinite limit[1] of an algebra is an algebraic construction that appears in the study of profinite groups. Much less is known about the profinite limit of lattice expansions, and that is what we will study in this report. The main goal is to better understand the relation between the profinite limit of an algebra and its MacNeille completion and canonical extension.

By lattice expansions we mean bounded distributive lattices with added operations. Most of the time we will be more specific though; to demonstrate our proof technique we first use modal algebras as an example.

We should explain our choice to speak of the profinite *limit* $\hat{\mathbb{A}}$ of an algebra $\mathbb{A}$. It is customary to call the algebraic construction we have in mind the profinite completion; however if we want completions to be extensions, we cannot always construct the profinite *completion*. The reason is that although we always have a natural mapping from $\mathbb{A}$ to $\hat{\mathbb{A}}$, that mapping is an embedding if and only if $\mathbb{A}$ is residually finite (this will be explained later on). While this may be a harmless condition in case $\mathbb{A}$ is just a distributive lattice, this need not be the case for lattice expansions. Therefore by speaking of the profinite completion we would limit our results to residually finite algebras, which is unnecessary. Thus we have adopted the phrase 'profinite limit' to indicate the construction $\hat{\mathbb{A}}$ regardless of whether $\mathbb{A}$ is residually finite.

Completions are our main interest though. One reason is that they may be used for completeness results for logics that have an algebraic semantics. Also, in the case of e.g. modal logic, where we have a duality theory linking algebraic semantics to Kripke semantics, the canonical extension (an algebraic completion) is used in the completeness theory for Kripke frames. For these and other reasons completions of lattice expansions are currently an active area of research (see [12], [8], [6], [7] for instance).

In articles such as [7], [11] and [3] the authors also study the relations *between* various kinds of completions. In [11] and [3] the relation between the profinite completion and the canonical extension is shown to be one of equality, under the right circumstances. What happens however when the circumstances are not right?

The profinite limit $\hat{\mathbb{A}}$ is constructed as a special subalgebra of a product of finite homomorphic images of $\mathbb{A}$. We would like to emphasize two important consequences of the fact that we can construct the profinite limit this way. First of all, this means

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[1]Our choice of words is non-standard, we will explain shortly.

that every identity that is valid on $\mathbb{A}$ is valid on $\hat{\mathbb{A}}$, quite unlike what happens with the canonical extension $\mathbb{A}^\sigma$ and the MacNeille completion $\bar{\mathbb{A}}$. Secondly, there are no parameters for us to tweak. When extending a lattice expansion $\mathbb{A}$ to e.g. the canonical extension, we actually have a choice between the upper and lower extension (see Section 2 for definitions). The reason for this is that when constructing the canonical extension of a lattice expansion, we first extend the lattice skeleton and we then get to choose how we extend the non-lattice operations. This is radically different from the construction of the profinite limit $\hat{\mathbb{A}}$ because there *all* operations on $\mathbb{A}$, be they part of the lattice structure or not, are extended from $\mathbb{A}$ to $\hat{\mathbb{A}}$.

So how might these three different constructions ($\hat{\mathbb{A}}$, $\mathbb{A}^\sigma$, $\bar{\mathbb{A}}$) interrelate? We first adopt an algebraic approach, which allows us to prove two negative results for modal algebras in Section 3: in general, the profinite limit $\hat{\mathbb{A}}$, the MacNeille completion $\bar{\mathbb{A}}$ and the canonical extension $\mathbb{A}^\sigma$ of a modal algebras are not the same.

Opposite the negative result about the canonical extension we can put two related positive results. Firstly, in [3] it is shown that in varieties of Heyting algebras, the profinite limit and the canonical extension are the same iff the variety is finitely generated. Moreover, in [11] it is shown that this can be generalized to varieties of monotone lattice expansions with a finite number of non-lattice operations. Since our modal algebras fall under the latter scope, our negative result gives an indirect proof that the variety of modal algebras is not finitely generated.

The methods of [11] are purely algebraic, like ours in Section 3 (excluding Lemma 3.16 where we use duality). In [3] (topological) duality theory is put to use. While trying to show when the profinite limit and the canonical extension of a Heyting algebra are equal, the authors of [3] establish using duality that the profinite limit is always a homomorphic image of the canonical extension. So even though the profinite limit and the canonical extension of a Heyting algebra are not always the same, they are always related.

The result of [3] that the profinite completion of a Heyting algebra is always a homomorphic image of the canonical extension inspired me to introduce some concepts from category theory which allow us to prove a similar positive result in Section 4 for modal algebras: the profinite limit $\hat{\mathbb{A}}$ of a modal algebra $\mathbb{A}$ is always a homomorphic image of the canonical extension $\mathbb{A}^\sigma$. The introduction of category theory pays off in Section 5 when we reflect on our proof methods from Section 4 and argue that they may be applied to distributive lattices with operators. At a certain level of generality our approach will break down though, because we can no longer rely on a nice duality theory. In Section 6 we will therefore develop a hybrid of our duality-based method and more traditional algebraic techniques, building on the work of [8] to further generalize our results from Section 5 so that they apply to arbitrary (distributive) lattice expansions. Finally we look at possible continuations of our line of investigation in Section 7.

2. Preliminaries

The results we will discuss in the sections below concern *completions* of *algebras*, and *categories of algebras* and *duality*. We assume that the reader is familiar with Kripke semantics for modal logic and has a basic familiarity with some form of algebra. The reader may consult Chapter 1 of [4] for an introduction to normal modal logics and Kripke semantics.

2.1. **Lattice expansions.**

For our purposes, a lattice is a bounded distributive lattice. A lattice expansion is a lattice with any number of added operations. For an introduction to the theory of universal algebra, we suggest [5].

Definition 2.1. A *(bounded) distributive lattice* is an algebra $\mathbb{A} = \langle A, \vee, \wedge, 0, 1\rangle$ with two binary operations $\vee$ (join) and $\wedge$ (meet), and two constants 0 and 1. It satisfies the following identities:

- the commutative laws $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$,
- the associative laws $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
- the absorption laws $x = x \vee (x \wedge y)$ and $x = x \wedge (x \vee y)$,
- 0 and 1 are neutral elements: $x \vee 0 = x$ and $x \wedge 1 = x$,
- the distributive laws $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (z \vee y) \wedge (x \vee z)$.

We can define a partial order on a lattice by setting $x \leq y$ iff $x \vee y = y$ (or equivalently iff $x \wedge y = x$). Note that with this ordering, $x \vee y$ and $x \wedge y$ are the least upper bound and greatest lower bound of $\{x, y\}$, respectively. A *homomorphism* $f\colon \mathbb{A} \to \mathbb{B}$ is a function $f\colon A \to B$ such that $f(0) = 0$, $f(1) = 1$, $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$, i.e. it is a function that commutes with the operations of the algebra.

Definition 2.2. A *filter* of a lattice $\mathbb{A}$ is a set $\nabla \subseteq \mathbb{A}$ such that

- $1 \in \nabla$,
- if $a \in \nabla$ and $a \leq b \in \mathbb{A}$, then $b \in \nabla$,
- if $a, b \in \nabla$ then $a \wedge b \in \nabla$.

An *ideal* $\Delta \subseteq \mathbb{A}$ is the order dual of a filter, i.e.

- $0 \in \Delta$,
- if $a \in \Delta$ and $a \geq b \in \mathbb{A}$, then $b \in \Delta$,
- if $a, b \in \Delta$ then $a \vee b \in \Delta$.

For any $a \in \mathbb{A}$, the *principal filter* generated by a is the set $a{\uparrow} = \{b \in \mathbb{A} \mid a \leq b\}$. A filter ∇ is *proper* if $0 \notin \nabla$. A filter ∇ is *prime* if it is proper and for all $a, b \in \mathbb{A}$ such that $a \vee b \in \nabla$, either $a \in \nabla$ or $b \in \nabla$.

Prime filters are an important part of the duality theory for distributive lattices, which we will see when we discuss categories below. First we introduce a special kind of distributive lattice which should be familiar to the reader who has seen some classical propositional logic.

Definition 2.3. We call an algebra $\mathbb{A} = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ a *Boolean algebra* if

- $\langle A, \vee, \wedge, 0, 1\rangle$ is a distributive lattice,
- $x \wedge \neg x = 0$ and $x \vee \neg x = 1$.

A Boolean algebra is *complete* if any set $\{a_i \mid i \in I\} \subseteq \mathbb{A}$ has a least upper bound $\bigvee_I a_i$ and a greatest lower bound $\bigwedge_I a_i$ with respect to the ordering $\leq$. In a Boolean algebra $\mathbb{A}$, we call $a \in \mathbb{A}$ an *atom* if for all $b \in \mathbb{A}$, $b < a$ implies $b = 0$. $\mathbb{A}$ is *atomic* if there is an atom below every non-zero $b \in \mathbb{A}$. A complete, atomic Boolean algebra is called *perfect*.

In a Boolean algebra, prime filters can be characterized in two alternative ways. Firstly, a filter of a Boolean algebra $\mathbb{A}$ is prime iff it is maximal. This means that if $\nabla \subseteq \mathbb{A}$ is a prime filter and $a \notin \nabla$, then there is no proper filter containing $\nabla \cup \{a\}$. Secondly, a filter of $\mathbb{A}$ is prime iff it is an ultrafilter, that is if for every $a \in \mathbb{A}$, either $a \in \nabla$ or $\neg a \in \nabla$.

We have chosen to present the first batch of our results for modal algebras in Sections 3 and 4, although most of them are valid in a wider setting. The extent of their generality will be discussed in Section 5.

Definition 2.4. $\mathbb{A} = \langle A, \vee, \wedge, \neg, \Diamond, 0, 1\rangle$ is a *modal algebra* if

- $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra,
- $\Diamond 0 = 0$ and $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$.

A complete modal algebra $\mathbb{A}$ is *completely additive* if for any set $\{a_i \mid i \in I\} \subseteq \mathbb{A}$, we have $\Diamond \bigvee_I a_i = \bigvee_I \Diamond a_i$, i.e. if $\Diamond$ not only commutes with finite joins but also with infinite joins. A complete, atomic, completely additive modal algebra is also called *perfect*.

We assume that the reader is familiar with the constructions of homomorphic images, subalgebras and (direct) products (if not, see [5]). We will give a definition for a construction that combines all three of these.

Definition 2.5. We say that $e\colon \mathbb{A} \hookrightarrow \prod_I \mathbb{B}_i$ is a subdirect embedding if for all $i \in I$, $\pi_i e\colon \mathbb{A} \twoheadrightarrow \mathbb{B}_i$, i.e. if every $\mathbb{B}_i$ is a homomorphic image not only of $\prod_I \mathbb{B}_i$ but also of $\mathbb{A}$ via the embedding $e\colon \mathbb{A} \hookrightarrow \prod_I \mathbb{B}_i$. If there exists a subdirect embedding $e\colon \mathbb{A} \hookrightarrow \prod_I \mathbb{B}_i$ where every $\mathbb{B}_i$ is finite, we call $\mathbb{A}$ *residually finite*.

2.2. **Categories of algebras and duality.** A category is a structure of objects and arrows between objects. In our case, the objects will be algebras and the arrows will be homomorphisms. We will be dealing with the following categories initially:

Definition 2.6. The objects of the category of modal algebras **MA** are all modal algebras. Its arrows (or 'morphisms') are all modal algebra homomorphisms, i.e. all functions $f\colon \mathbb{A} \to \mathbb{B}$ which commute with $\vee$, $\wedge$, $\neg$, $\Diamond$, 0 and 1. The category of perfect modal algebras $\mathbf{MA}^+$ has all perfect modal algebras as objects and complete homomorphisms between perfect algebras (see Definition 2.11 below) as morphisms.

One reason for introducing the added abstraction of categories is that the duality theory for modal algebras can be described well using categories. Duality is what connects (perfect) modal algebras to Kripke frames. The connection is quite rich, but we present only what we will use. For a better overview, see [13].

Definition 2.7. Given a modal algebra $\mathbb{A}$ we construct its *ultrafilter frame* $\mathbb{A}_\bullet$ by taking the set of ultrafilters of $\mathbb{A}$ as the points in our frame, and saying that $\nabla_1 R \nabla_2$ iff for all $a \in \nabla_2$, $\Diamond a \in \nabla_1$. If $\mathbb{A}$ is a perfect modal algebra, we can also construct its *atom structure* $\mathbb{A}_+$ by taking the set of atoms as the points in our frame and defining aRb iff $a \leq \Diamond b$.

Conversely, if we have a Kripke frame $\mathfrak{F} = \langle W, R\rangle$ we may construct its *complex algebra* $\mathfrak{F}^+ = \langle \mathcal{P}(W), \cup, \cap, (\cdot)^c, m_R, \emptyset, W\rangle$, where $(\cdot)^c$ is set-theoretic complementation relative to W and $m_R(X) = \{w \in W \mid wRv \in X\}$.

Now $\mathbb{A}$ is a perfect modal algebra iff $(\mathbb{A}_+)^+ \cong \mathbb{A}$, and if $\mathbb{A}$ is finite, then $\mathbb{A}_\bullet \cong \mathbb{A}_+$. For those familiar with Kripke semantics, it may be worth noting that $(\mathfrak{F}^+)_\bullet$ is the ultrafilter extension of a Kripke frame $\mathfrak{F}$ and that if $\mathbb{A}$ is the Lindenbaum-Tarski algebra of a normal logic Λ, then $(\mathbb{A})_\bullet$ is (isomorphic to) the canonical frame for Λ.

So we have a connection between modal algebras and Kripke frames; why do we need categories? Because the connection extends to homomorphisms between algebras and bounded morphisms between frames:

Definition 2.8. Let $f\colon \mathbb{A} \to \mathbb{B}$ be a morphism of **MA**. Then $f_\bullet\colon \nabla \mapsto f^{-1}(\nabla)$ is a bounded morphism from $\mathbb{B}_\bullet$ to $\mathbb{A}_\bullet$. If $f\colon \mathbb{A} \to \mathbb{B}$ is a morphism of $\mathbf{MA}^+$, then $f_+\colon b \mapsto \bigwedge\{a \in \mathbb{A} \mid b \leq f(a)\}$ is a bounded morphism from $\mathbb{B}_+$ to $\mathbb{A}_+$. Note how in both cases the arrows are reversed.

Conversely, if $g\colon \mathfrak{F} \to \mathfrak{G}$ is a bounded morphism between Kripke frames, then $g^+\colon X \mapsto g^{-1}(X)$ is a morphism of $\mathbf{MA}^+$ from $\mathfrak{G}^+$ to $\mathfrak{F}^+$, i.e. a complete homomorphism between perfect modal algebras.

There are several preservation results for $(\cdot)_\bullet$, $(\cdot)_+$ and $(\cdot)^+$ applied to morphisms, although maybe we should not say preservation, as the arrows are reversed. We state a few:

Lemma 2.9. *Let $f\colon \mathbb{A} \to \mathbb{B}$ be a homomorphism between modal algebras, let $g\colon \mathbb{C} \to \mathbb{D}$ be a complete homomorphism between perfect modal algebras and let $g\colon \mathfrak{F} \to \mathfrak{G}$ be a bounded morphism. Then we have the following:*

(1) $f_\bullet\colon \mathbb{B}_\bullet \hookrightarrow \mathbb{A}_\bullet$ *is an embedding if* $f\colon \mathbb{A} \twoheadrightarrow \mathbb{B}$ *is a surjection,*
(2) $g_+\colon \mathbb{D}_+ \hookrightarrow \mathbb{C}_+$ *is an embedding if* $g\colon \mathbb{C} \twoheadrightarrow \mathbb{D}$ *is a surjection,*
(3) $h^+\colon \mathfrak{G}^+ \twoheadrightarrow \mathfrak{F}^+$ *is a surjection if* $h\colon \mathfrak{F} \hookrightarrow \mathfrak{G}$ *is an embedding.*

Definition 2.10. By **KFr** we denote the category of Kripke frames and bounded morphisms.

Thus, $(\cdot)_\bullet$ and $(\cdot)_+$ somehow map **MA** and **MA**$^+$ to **KFr** respectively, and $(\cdot)^+$ maps **KFr** to **MA**$^+$, but in all three cases arrows are reversed. Such an arrow-reversing map between categories is called a *contravariant functor*. Recall that, as we remarked above, $(\mathbb{A}_+)^+ \cong \mathbb{A}$ iff $\mathbb{A}$ is perfect. Something similar is the case for complete homomorphisms $f\colon \mathbb{A} \to \mathbb{B}$ between perfect modal algebras. Let $\epsilon_\mathbb{A}\colon \mathbb{A} \to (\mathbb{A}_+)^+$ be the isomorphism witnessing that $(\mathbb{A}_+)^+ \cong \mathbb{A}$, i.e. $\epsilon_\mathbb{A}\colon a \mapsto \{b \in \mathbb{A}_+ \mid b \leq a\}$. Then $(f_+)^+\epsilon_\mathbb{A} = \epsilon_\mathbb{B} f$, or equivalently, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "\epsilon_\mathbb{A}"] \arrow[d, "f"'] & (\mathbb{A}_+)^+ \arrow[d, "(f_+)^+"] \\
\mathbb{B} \arrow[r, "\epsilon_\mathbb{B}"'] & (\mathbb{B}_+)^+
\end{tikzcd}$$

We see that f and $(f_+)^+$ are in a sense the same, even though their domains and codomains are not. Because we also have $\mathfrak{F} \cong (\mathfrak{F}^+)_+$ and the same picture as above for bounded morphisms $g\colon \mathfrak{F} \to \mathfrak{G}$, we say that **MA**$^+$ and **KFr** are *dually equivalent*, that is they are in a sense the same, if we keep in mind that arrows are reversed. For a definition of dual equivalence of categories, see [1].

2.3. **Completions.** In a lattice $\mathbb{A}$, any finite collection of elements $a_0, \dots, a_n \in \mathbb{A}$ has a least upper bound $(\cdots(a_0 \vee a_1) \vee \cdots) \vee a_n) = \bigvee_{i \leq n} a_i$ and greatest lower bound $\bigwedge_{i \leq n} a_i$.

Definition 2.11. A lattice or lattice expansion $\mathbb{A}$ is *complete* if any collection of elements has a join and a meet. A completion of $\mathbb{A}$ is a pair $\langle e, \mathbb{B}\rangle$ such that $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ is an embedding and $\mathbb{B}$ is complete.

A *complete homomorphism* between complete algebras is a homomorphism $f\colon \mathbb{A} \to \mathbb{B}$ such that for all $\{a_i \mid i \in I\} \subseteq \mathbb{A}$, $f(\bigvee_I a_i) = \bigvee_I f(a_i)$ and $f(\bigwedge_I a_i) = \bigwedge_I f(a_i)$, i.e. f not only commutes with finite joins and meets, but also with infinite joins and meets. We say that $\mathbb{A}$ is a regular subalgebra of $\mathbb{B}$ if the identity embedding is a complete homomorphism.

Given a completion $\langle e, \mathbb{B}\rangle$ of $\mathbb{A}$ we may distinguish the *open* and *closed* elements of $\mathbb{B}$:

$$O_\mathbb{A}(\mathbb{B}) = \big\{\bigvee e[\Delta] \mid \Delta \text{ is an ideal of } \mathbb{A}\big\},$$
$$K_\mathbb{A}(\mathbb{B}) = \big\{\bigwedge e[\nabla] \mid \nabla \text{ is a filter of } \mathbb{A}\big\}.$$

Definition 2.12. Let $\langle e, \mathbb{B}\rangle$ be a completion of a lattice $\mathbb{A}$. We define the following properties of $\langle e, \mathbb{B}\rangle$:

- we say $\langle e, \mathbb{B}\rangle$ is a *dense* extension if for all $b \in \mathbb{B}$ we have both

$$b = \bigvee \big\{k \in K_\mathbb{A}(\mathbb{B}) \mid k \leq b\big\}$$

and

$$b = \bigwedge \big\{u \in O_\mathbb{A}(\mathbb{B}) \mid b \leq u\big\},$$

i.e. every element of $\mathbb{B}$ is a join of closed elements and a meet of open elements,

- $\langle e, \mathbb{B}\rangle$ is *join-dense* if for all $b \in \mathbb{B}$, $b = \bigvee\{e(a) \in \mathbb{A} \mid e(a) \leq b\}$, i.e. if every element of $\mathbb{B}$ is open (*meet-density* is defined analogously),
- $\langle e, \mathbb{B}\rangle$ is *compact* if for every filter ∇ and ideal Δ of $\mathbb{A}$ such that $\nabla \cap \Delta = \emptyset$, we have $\bigwedge e[\nabla] \not\leq \bigvee e[\Delta]$ (where the meets and joins are taken in $\mathbb{B}$).

The unique (up to isomorphism) completion $\langle e, \mathbb{B}\rangle$ of $\mathbb{A}$ that is both join-dense and meet-dense is called the *MacNeille completion* $\bar{\mathbb{A}}$. The unique (up to isomorphism) completion $\langle e, \mathbb{B}\rangle$ of $\mathbb{A}$ that is both dense and compact is called the *canonical extension* $\mathbb{A}^\sigma$.

We will sometimes use the following alternative definition of $\langle e, \mathbb{B}\rangle$ being a dense extension of $\mathbb{A}$: for all $b \in \mathbb{B}$, we have both

$$b = \bigvee \left\{ \bigwedge e[\nabla] \mid \bigwedge e[\nabla] \leq b \text{ and } \nabla \text{ is a filter of } \mathbb{A} \right\}$$

and

$$b = \bigwedge \left\{ \bigvee e[\Delta] \mid \bigvee e[\Delta] \geq b \text{ and } \Delta \text{ is an ideal of } \mathbb{A} \right\}.$$

Instead of calling $\langle e, \mathbb{B}\rangle$ a dense (or compact, join-dense, etc.) extension of $\mathbb{A}$, we sometimes call $e \colon \mathbb{A} \hookrightarrow \mathbb{B}$ a dense embedding.

Note that above we only defined the canonical extension and MacNeille completion of a lattice. When it comes to defining them for lattices with added operations, we have several options. We present two for the canonical extension:

Definition 2.13. We consider a lattice expansion $\mathbb{A}$ with one unary operation $\Diamond$. We will define $\Diamond^\sigma$ and $\Diamond^\pi$, the *lower* and *upper* extension of $\Diamond$, respectively. First we define the extensions on closed and open elements respectively:

$$\Diamond^\sigma\left(\bigwedge e[\nabla]\right) = \bigwedge \left\{ e(\Diamond a) \mid a \in \nabla \right\} = \bigwedge e\Diamond[\nabla],$$
$$\Diamond^\pi\left(\bigvee e[\Delta]\right) = \bigvee \left\{ e(\Diamond a) \mid a \in \Delta \right\} = \bigvee e\Diamond[\Delta],$$

for $\bigwedge e[\nabla]$ and $\bigvee e[\Delta]$ arbitrary closed and open elements of $\mathbb{A}^\sigma$, respectively. Building on that we define

$$\Diamond^\sigma(x) = \bigvee \left\{ \Diamond^\sigma(k) \mid x \geq k \in K(\mathbb{A}^\sigma) \right\},$$
$$\Diamond^\pi(x) = \bigwedge \left\{ \Diamond^\pi(u) \mid x \leq u \in O(\mathbb{A}^\sigma) \right\}.$$

In case $\Diamond^\sigma = \Diamond^\pi$ we say that $\Diamond$ is *smooth*, in which case we will default to writing $\Diamond^\sigma$. We can make the same construction for operations with arities greater than one. If $\mathbb{A}$ were a lattice expansion with more than one added operation, we could choose to use the lower or upper extension for each individual operation. If there is only one operation however, or if we choose uniformly, we write $\mathbb{A}^\sigma$ for the lower extension of $\mathbb{A}$ and $\mathbb{A}^\pi$ for the upper extension of $\mathbb{A}$.

Finally we present an alternative way of obtaining the canonical extension of a modal algebra. The $\Diamond$ of a modal algebra is always smooth, so we can simply write $\mathbb{A}^\sigma$ (for either the lower or the upper extension). Surprisingly, $\mathbb{A}^\sigma \cong (\mathbb{A}_\bullet)^+$. That is, the *(covariant) functor* $(\cdot)^\sigma = ((\cdot)_\bullet)^+ \colon \mathbf{MA} \to \mathbf{KFr} \to \mathbf{MA}^+$ gives us the canonical extension. A covariant functor is a functor that does not reverse arrows.

3. The profinite limit of a modal algebra

We will construct the profinite limit $\hat{\mathbb{A}}$ of a modal algebra $\mathbb{A}$ without using duality. We then compare the profinite limit $\hat{\mathbb{A}}$ to the MacNeille completion $\bar{\mathbb{A}}$ and the canonical extension $\mathbb{A}^\sigma$.

3.1. **The construction.** We start our construction of $\hat{\mathbb{A}}$ with a collection of quotients of $\mathbb{A}$, or more precisely, a set of congruences. By $\operatorname{Con}\mathbb{A}$ we denote the lattice of congruences of $\mathbb{A}$.

Definition 3.1. We define a set of congruences

$$\Phi = \bigl\{\theta \in \operatorname{Con}\mathbb{A} \mid \mathbb{A}/\theta \text{ is finite}\bigr\}.$$

These congruences index the finite quotients of $\mathbb{A}$.

Lemma 3.2. *Our index set Φ is a sublattice of* $\operatorname{Con}\mathbb{A}$ *under the ordering $\subseteq$.*

Proof. Let $\theta, \psi \in \Phi$. Then $\theta \subseteq \theta \vee \psi$, thus $\mathbb{A}/\theta \twoheadrightarrow \mathbb{A}/(\theta \vee \psi)$, so that $|\mathbb{A}/(\theta \vee \psi)| \leq |\mathbb{A}/\theta| < \omega$, whence $\theta \vee \psi \in \Phi$.

Furthermore, $\mathbb{A}/(\theta \vee \psi)$ can be embedded in $\mathbb{A}/\theta \times \mathbb{A}/\psi$ using the mapping $f\colon a/(\theta \wedge \psi) \mapsto \langle a/\theta, a/\psi\rangle$. First we show that f is well-defined. Suppose that $a/(\theta \wedge \psi) = b/(\theta \wedge \psi)$, then since $\theta \wedge \psi = \theta \cap \psi$, it follows that $a/\theta = b/\theta$ and $a/\psi = b/\psi$, whence $f\bigl(a/(\theta \wedge \psi)\bigr) = f\bigl(b/(\theta \wedge \psi)\bigr)$. Secondly, the reader may verify that f is a homomorphism. Finally we show that f is injective: if $f\bigl(a/(\theta \wedge \psi)\bigr) = f\bigl(b/(\theta \wedge \psi)\bigr)$, then $a/\theta = b/\theta$ and $a/\psi = b/\psi$, whence $(a, b) \in \theta \cap \psi = \theta \wedge \psi$. Thus we have shown that $\mathbb{A}/(\theta \wedge \psi) \hookrightarrow \mathbb{A}/\theta \times \mathbb{A}/\psi$, whence $|\mathbb{A}/(\theta \wedge \psi)| \leq |\mathbb{A}/\theta \times \mathbb{A}/\psi| = |\mathbb{A}/\theta| \times |\mathbb{A}/\psi| < \omega$. It follows that $\theta \wedge \psi \in \Phi$. $\square$

Note that Φ is not a bounded sublattice of $\operatorname{Con}\mathbb{A}$ if $|\mathbb{A}| \geq \omega$, i.e. it does not inherit the top and bottom elements of $\operatorname{Con}\mathbb{A}$. After all, the bottom element of $\operatorname{Con}\mathbb{A}$ is the identity congruence $\mathrm{Id}_{\mathbb{A}} = \{(a, a) \mid a \in \mathbb{A}\}$, thus $\mathbb{A}/\mathrm{Id} \cong \mathbb{A}$. It follows that $\mathbb{A}/\mathrm{Id}$ is infinite if $\mathbb{A}$ is infinite, so Id, the bottom element of $\operatorname{Con}\mathbb{A}$ cannot be the bottom element of Φ.

Definition 3.3. We define the diagram F, consisting of all quotients $\mathbb{A}/\theta$ for $\theta \in \Phi$, connected by homomorphisms $\varphi_{\theta\theta'}\colon \mathbb{A}/\theta \twoheadrightarrow \mathbb{A}/\theta'$ for $\theta, \theta' \in \Phi$ such that $\theta \subseteq \theta'$, where

$$\varphi_{\theta\theta'}\colon a/\theta \mapsto a/\theta'.$$

In other words, the $\varphi_{\theta\theta'}\colon \mathbb{A}/\theta \twoheadrightarrow \mathbb{A}/\theta'$ are natural maps. (See Figure 1.) F is in fact a functor, as we will see in Section 4 below.

Figure 1 is a bit unusual in that it depicts a lattice as growing to the right, instead of growing upwards, as is customary. Note that the natural maps $\varphi_{\theta\theta'}\colon \mathbb{A}/\theta \twoheadrightarrow \mathbb{A}/\theta'$ are all complete homomorphisms, because all homomorphisms between finite algebras are complete.

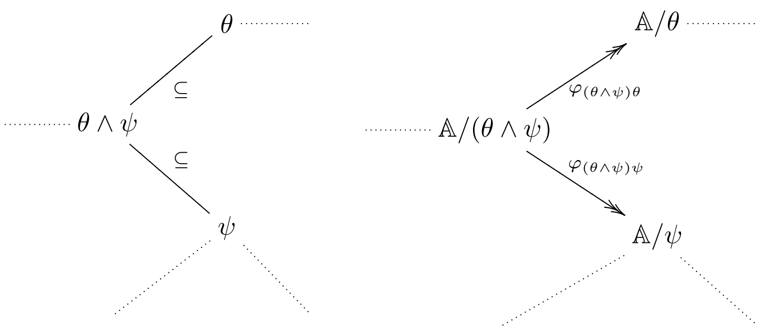

FIGURE 1. A fragment of the lattice Φ and the diagram F it indexes.

By $\prod_\Phi \mathbb{A}/\theta$ we denote the product of all finite quotients of $\mathbb{A}$, where the operations on the product are defined coordinate-wise as usual. Note the immediate consequence that $\prod_\Phi \mathbb{A}/\theta$ is complete, as all finite lattice expansions are complete and products of complete lattice expansions are complete. By $\pi_\theta \colon \prod_\Phi \mathbb{A}/\theta \twoheadrightarrow \mathbb{A}/\theta$ we denote the projections associated with the product. Also note that all π_θ are complete homomorphisms.

Definition 3.4. We define $\hat{\mathbb{A}}$, the profinite limit of $\mathbb{A}$:

$$\hat{\mathbb{A}} = \bigl\{\alpha \in \textstyle\prod_\Phi \mathbb{A}/\theta \mid \text{for all } \theta, \theta \in \Phi \text{ s.t. } \theta \subseteq \theta',\ \varphi_{\theta\theta'}\pi_\theta(\alpha) = \pi_{\theta'}(\alpha)\bigr\}.$$

Lemma 3.5. *$\hat{\mathbb{A}}$ is a complete regular subalgebra of $\prod_\Phi \mathbb{A}/\theta$.*

Proof. We will first show that $\hat{\mathbb{A}}$ is a subalgebra of $\prod_\Phi \mathbb{A}/\theta$. Let $\alpha_0, \dots, \alpha_{n-1} \in \hat{\mathbb{A}}$ and let $\bigstar$ be an operation of $\mathbb{A}$ of arity n. Let $\theta, \theta' \in \Phi$ such that $\theta \subseteq \theta'$, then

$$\begin{aligned}\varphi_{\theta\theta'}\pi_\theta\bigl(\bigstar(\alpha_0, \dots, \alpha_{n-1})\bigr) &= \bigstar\bigl(\varphi_{\theta\theta'}\pi_\theta(\alpha_0), \dots, \varphi_{\theta\theta'}\pi_\theta(\alpha_{n-1})\bigr)\\ &= \bigstar\bigl(\pi_{\theta'}(\alpha_0), \dots, \pi_{\theta'}(\alpha_{n-1})\bigr) = \pi_{\theta'}\bigl(\bigstar(\alpha_0, \dots, \alpha_{n-1})\bigr),\end{aligned}$$

where the first and third equality follow from the fact that $\varphi_{\theta\theta'}$, π_θ and $\pi_{\theta'}$ are homomorphisms and the second equality follows from the fact that $\alpha_0, \dots, \alpha_{n-1} \in \hat{\mathbb{A}}$. It follows that $\bigstar(\alpha_0, \dots, \alpha_{n-1}) \in \hat{\mathbb{A}}$, whence $\hat{\mathbb{A}}$ is a subalgebra of $\prod_\Phi \mathbb{A}/\theta$.

Secondly, we will show that if $\{\alpha_i\}_{i\in I} \subseteq \hat{\mathbb{A}}$ for any index set I, then $\bigvee_I \alpha_i$, the join in $\prod_\Phi \mathbb{A}/\theta$, is an element of $\hat{\mathbb{A}}$. Let $\theta, \theta' \in \Phi$ such that $\theta \subseteq \theta'$. Because π_θ and $\varphi_{\theta\theta'}$ are complete, we find that $\varphi_{\theta\theta'}\pi_\theta \bigvee_I \alpha_i = \bigvee_I \varphi_{\theta\theta'}\pi_\theta\alpha_i$. Because $\{\alpha_i\}_{i\in I} \subseteq \hat{\mathbb{A}}$, we find that $\bigvee_I \varphi_{\theta\theta'}\pi_\theta\alpha_i = \bigvee_I \pi_{\theta'}\alpha_i$. Since $\pi_{\theta'}$ is also complete, it follows that $\varphi_{\theta\theta'}\pi_\theta \bigvee_I \alpha_i = \pi_{\theta'} \bigvee_I \alpha_i$, thus $\bigvee_I \alpha_i \in \hat{\mathbb{A}}$. The proof for $\bigwedge$ is analogous. Therefore $\hat{\mathbb{A}}$ is a complete regular subalgebra of $\prod_\Phi \mathbb{A}/\theta$. $\square$

We introduce some notation. Observe that if $(f_\theta \colon \mathbb{B} \to \mathbb{A}/\theta)_{\theta\in\Phi}$ is a family of homomorphisms, we may define $f \colon b \mapsto \bigl(f_\theta(b)\bigr)_{\theta\in\Phi}$ and obtain a homomorphism $f \colon \mathbb{B} \to \prod_\Phi \mathbb{A}/\theta$. Conversely, if we have a homomorphism $f \colon \mathbb{B} \to \prod_\Phi \mathbb{A}/\theta$, then we may write $f_\theta = \pi_\theta f$ for all $\theta \in \Phi$, to get a family of maps $(f_\theta \colon \mathbb{B} \to \mathbb{A}/\theta)_{\theta\in\Phi}$. In other words, we may see a family of maps sharing one domain as a 'bundle' of maps to the product of their codomains, and conversely a map to a product may be seen as a bundle of maps to the factors of the product. We will change perspectives between f and $(f_\theta)_{\theta\in\Phi}$ whenever we find it convenient.

Definition 3.6. We denote the identity embedding witnessing Lemma 3.5 by $\hat{\pi} \colon \hat{\mathbb{A}} \hookrightarrow \prod_\Phi \mathbb{A}/\theta$. Moreover, we define $\mu \colon \mathbb{A} \to \prod_\Phi \mathbb{A}/\theta$ as follows:

$$\mu \colon a \mapsto (a/\theta)_{\theta\in\Phi}.$$

Note that $\hat{\pi}_\theta = \pi_\theta \restriction \hat{\mathbb{A}}$.

Lemma 3.7. *The range of $\mu \colon \mathbb{A} \to \prod_\Phi \mathbb{A}/\theta$ lies in $\hat{\mathbb{A}}$, i.e. $\mu \colon \mathbb{A} \to \hat{\mathbb{A}}$.*

Proof. Let $a \in \mathbb{A}$. Now $\mu(a) \in \hat{\mathbb{A}}$ if for all $\theta, \theta' \in \Phi$ such that $\theta \subseteq \theta'$, we have $\varphi_{\theta\theta'}\pi_\theta\mu(a) = \pi_{\theta'}\mu(a)$. However,

$$\varphi_{\theta\theta'}\pi_\theta\mu(a) = \varphi_{\theta\theta'}\mu_\theta(a) = \varphi_{\theta\theta'}(a/\theta) = a/\theta' = \mu_{\theta'}(a) = \pi_{\theta'}\mu(a).$$

It follows that $\mu(a) \in \hat{\mathbb{A}}$. $\square$

Now we have a complete algebra $\hat{\mathbb{A}}$ and a homomorphism $\mu \colon \mathbb{A} \to \hat{\mathbb{A}}$. But is $\mu \colon \mathbb{A} \to \hat{\mathbb{A}}$ a completion? We attribute the following Lemma to folklore.

Lemma 3.8. *The homomorphism $\mu \colon \mathbb{A} \to \hat{\mathbb{A}}$ is an embedding iff $\mathbb{A}$ is residually finite.*

Proof. First we observe that every μ_θ is a surjection, as for any $a/\theta \in \mathbb{A}/\theta$ with $\theta \in \Phi$, we have $\mu_\theta(a) = a/\theta$. Now for the left to right direction of our Lemma: suppose that μ is injective. Then $\hat{\pi}\mu\colon \mathbb{A} \hookrightarrow \prod_\Phi \mathbb{A}/\theta$, where $\pi_\theta\hat{\pi}\mu = \hat{\pi}_\theta\mu = \pi_\theta\mu = \mu_\theta$ is surjective for every $\theta \in \Phi$ and every $\mathbb{A}/\theta$ is finite, whence $\hat{\pi}\mu\colon \mathbb{A} \hookrightarrow \prod_\Phi \mathbb{A}/\theta$ is a subdirect embedding and $\mathbb{A}$ is therefore residually finite.

Conversely, if $f\colon \mathbb{A} \hookrightarrow \prod_X \mathbb{A}_x$ is a subdirect embedding and every $\mathbb{A}_x$ is finite, then we may define $\Psi = \{\ker f_x \mid x \in X\}$. Observe that $\Psi \subseteq \Phi$. As $\prod_X \mathbb{A}_x \cong \prod_\Psi \mathbb{A}/\theta$, we find a new subdirect embedding $(\mu_\theta)_{\theta\in\Psi}\colon \mathbb{A} \hookrightarrow \prod_\Psi \mathbb{A}/\theta$. But then $\mu\colon \mathbb{A} \to \prod_\Phi \mathbb{A}/\theta$ is also an embedding, i.e. μ is injective. □

This should explain why we are hesitant to call $\hat{\mathbb{A}}$ a completion, as that would require that it actually extends $\mathbb{A}$, which is not always the case.

3.2. **What kind of modal algebra is $\hat{\mathbb{A}}$?** Above we have almost ignored the fact that $\mathbb{A}$ and $\hat{\mathbb{A}}$ are modal algebras. Below we will investigate $\hat{\mathbb{A}}$ as a modal algebra.

Lemma 3.9. *Let $\mathbb{B}$ be an atomic modal algebra and let $\mathbb{A}$ be a complete regular subalgebra of $\mathbb{B}$. Then $\mathbb{A}$ is atomic.*

Proof. Let $a \in \mathbb{A}$ with $a > 0$, then we have to show that there is an atom below a in $\mathbb{A}$. Let $b \in \mathbb{B}$ be the atom below a in $\mathbb{B}$ and define

$$A_b := \{c \in \mathbb{A} \mid b \leq c\}.$$

Observe that $A_b \neq \emptyset$ as $a \in A_b$. Let $d = \bigwedge A_b$, then $d \in \mathbb{A}$. Towards a contradiction, assume that $d = 0$. Because $b \leq c$ for all $c \in A_b$, we get $b \leq \bigwedge A_b = d$, so $d = 0$ would imply $b = 0$, contradicting the fact that b is an atom. We conclude that $d > 0$. We claim that d is an atom of $\mathbb{A}$. Towards a contradiction, suppose that there is $d' \in \mathbb{A}$ such that $0 < d' < d$. If $b \leq d'$, it follows that $d' \in A_b$, whence $d \leq d'$, which is a contradiction. If on the other hand $b \not\leq d'$, we get that $b \leq \neg d'$ because b is an atom of $\mathbb{B}$, whence $d \leq \neg d'$, so that $d' \leq \neg d'$, whence $d' = 0$, which also is a contradiction. We conclude that there can be no such d'. It follows that d is an atom, so since $d \leq a$, we have proved that $\mathbb{A}$ is atomic. □

Corollary 3.10. *$\hat{\mathbb{A}}$ is atomic.*

Proof. Since a product of atomic algebras is atomic, $\prod_\Phi \mathbb{A}/\theta$ is atomic. The statement now follows from Lemmas 3.5 and 3.9. □

Lemma 3.11. *Let $\mathbb{B}$ be completely additive, let $\mathbb{A}$ be complete and let $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ be a complete embedding. Then $\mathbb{A}$ is completely additive.*

Proof. We will show that for any $\{a_i\}_{i\in I} \subseteq \mathbb{A}$, we have $\Diamond \bigvee_I a_i = \bigvee_I \Diamond a_i$. Let $\{a_i\}_{i\in I} \subseteq \mathbb{A}$, then we find that

$$e\big(\Diamond \bigvee_I a_i\big) = \Diamond \bigvee_I e(a_i).$$

Because $\mathbb{B}$ is completely additive, it follows that

$$\Diamond \bigvee_I e(a_i) = \bigvee_I \Diamond e(a_i) = e\Big(\bigvee_I \Diamond a_i\Big),$$

where the second equality again follows from the fact that e is a complete embedding. We therefore find that $e\big(\Diamond \bigvee_I a_i\big) = e\big(\bigvee_I \Diamond a_i\big)$, so since e is an embedding, it must be the case that $\Diamond \bigvee_I a_i = \bigvee_I \Diamond a_i$. We conclude that $\mathbb{A}$ is completely additive. □

Corollary 3.12. *$\hat{\mathbb{A}}$ is completely additive.*

Proof. A product of completely additive modal algebras is again completely additive, so by Lemmas 3.5 and 3.11, it follows that $\hat{\mathbb{A}}$ is completely additive. □

Theorem 3.13. *The profinite limit of a modal algebra is perfect.*

Proof. We only need to combine Lemma 3.5 and the two Corollaries above. □

3.3. **How does $\hat{\mathbb{A}}$ relate to other completions?** Now that we have a basic understanding of the construction of the profinite completion, we may investigate how this new construct relates to completions we already know. The knowledge that $\hat{\mathbb{A}}$ is always atomic immediately leads to a negative result, for which we rely on the following fact (see Ch. XII.3 of [2]):

Fact 3.14. *Let $\mathbb{A}$ be a Boolean algebra (or a lattice expansion containing a Boolean algebra). Then $\mathbb{A}$ is atomic iff $\bar{\mathbb{A}}$ (the MacNeille completion of $\mathbb{A}$) is atomic.*

Corollary 3.15. *There are modal algebras $\mathbb{A}$ such that the profinite limit and the MacNeille completion of $\mathbb{A}$ do not coincide, i.e. such that $\hat{\mathbb{A}} \ncong \bar{\mathbb{A}}$.*

So what about the canonical extension? Here we will use the Lindenbaum-Tarski algebra of the modal logic **K** (see [4] Ch. 5.2), which we will denote by $\mathbb{K}$, to prove another negative result. In universal algebra, $\mathbb{K}$ is called the *free algebra* over ω generators for the variety of modal algebras. One important property of the free modal algebra over ω generators is the fact that any finite modal algebra $\mathbb{B}$ is a homomorphic image of $\mathbb{K}$, i.e. there is some congruence θ of $\mathbb{K}$ such that $\mathbb{K}/\theta \cong \mathbb{B}$ for any finite modal algebra $\mathbb{B}$. Another thing worth noting is that the free modal algebra is residually finite since the modal logic **K** has the finite model property. Therefore we know $\langle \mu, \hat{\mathbb{K}} \rangle$ to be a completion of $\mathbb{K}$. Recall that compact extensions were defined in Definition 2.12. In the present case, where we have a Boolean algebra we might as well rephrase that definition as requiring that no proper filter ∇ of $\mathbb{A}$ has the property that $\bigwedge \mu[\nabla] = 0$ in $\hat{\mathbb{A}}$.

Lemma 3.16. *$\langle \mu, \hat{\mathbb{K}} \rangle$ is not a compact extension of $\mathbb{K}$.*

Proof. We will describe a proper filter ∇ of $\mathbb{K}$ for which $\bigwedge \mu[\nabla] = 0$, showing that $\langle \mu, \hat{\mathbb{K}} \rangle$ is not a compact extension of $\mathbb{K}$. For this proof we will exploit insights offered by Kripke frames using the fact below. Recall that $\mathfrak{F}^+$ is the complex algebra of $\mathfrak{F}$.

Fact 3.17. *Let δ be a closed formula (or equivalently a closed term) and let $\mathfrak{F} = \langle W, R \rangle$ be a Kripke frame. Then the following are equivalent:*

(1) *There is a $w \in W$ such that $\mathfrak{F}, w \Vdash \delta$,*
(2) *$\mathfrak{F}^+ \models \delta > 0$.*

In words, if a formula/term is satisfiable in a Kripke frame then that formula/term does not evaluate to zero in the complex algebra of that Kripke frame, and vice versa.

We define the following formulas for $n \geq 0$:

$$\gamma_n \equiv \Diamond(\Diamond^n 1 \wedge \Box^{n+1} 0),$$

and set $\Gamma = \{\gamma_n \mid n \in \omega\}$. A short argument using Kripke semantics will show that $\mathfrak{F}, w \Vdash \gamma_n$ iff there is a v_n with wRv_n such that there is an R-path of length n starting from v_n and there is no path of length $n+1$ starting from v_n. We call v_n an n-successor of w. Note that if $n \neq m$, no point s can simultaneously be an n- and an m-successor of w.

We will show that Γ has the finite meet property, i.e. that any finite subset of Γ has a non-zero meet. Take $\Gamma' \subseteq \Gamma$ such that $|\Gamma'| = n+1 < \omega$, then $\Gamma' = \{\gamma_{i_0}, \dots \gamma_{i_n}\}$ for certain $i_0, \dots, i_n \in \omega$. If we let $\mathfrak{F}$ be the disjoint union of paths of length i_j for all $0 \leq j \leq n$ with a root added below the first element of every path as in Figure 2, then we see that $\mathfrak{F}, w \Vdash \bigwedge \Gamma'$. It follows from the Fact above that $\mathfrak{F}^+ \models \bigwedge \Gamma' > 0$,

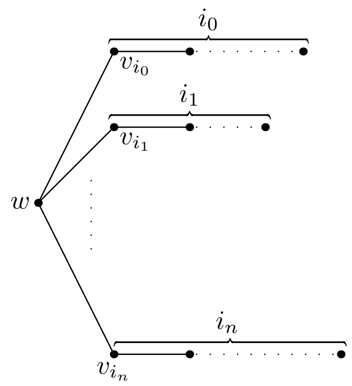

FIGURE 2. A frame for Γ'.

i.e. that if we interpret the sentences in Γ' as terms on the complex algebra of $\mathfrak{F}$, then the meet of those terms is not zero. Because $\mathfrak{F}$ is finite, so is $\mathfrak{F}^+$. But then because $\mathbb{K}$ is the free algebra over ω generators, it follows that $\mathbb{K} \twoheadrightarrow \mathfrak{F}^+$, thus $\mathbb{K} \models \bigwedge \Gamma' > 0$. It follows that Γ has the finite meet property, thus there must exist a proper filter $\nabla \supseteq \Gamma$ in $\mathbb{K}$. Towards a contradiction assume that $\bigwedge \mu[\nabla] > 0$ (we remind the reader that the latter meet is taken in $\hat{\mathbb{A}}$). It follows that $\bigwedge \mu[\Gamma] > 0$, thus there is some $\theta \in \Phi$ such that $\pi_\theta \bigwedge \mu[\Gamma] = \bigwedge \mu_\theta[\Gamma] > 0$, i.e. $\mathbb{K}/\theta \models \gamma_n > 0$ for every $n \in \omega$. Since $\mathbb{K}/\theta$ is finite, there must be some finite $\mathfrak{F}_\theta$ such that $\mathfrak{F}_\theta^+ \cong \mathbb{K}/\theta$. Again by the Fact above, it follows that there must be some point $w \in \mathfrak{F}_\theta$ which has a n-successor for every $n \in \omega$. In other words, w has infinitely many n-successors, so since $\mathfrak{F}_\theta$ is finite, not all n-successors of w are different, i.e. there must be an $s \in \mathfrak{F}_\theta$ that is both an n- and an m-successor for $m \neq n$ and $m, n \in \omega$. This contradicts our observation about the properties of n-successors above, so we conclude that $\bigwedge \mu[\nabla] = 0$, thus $\langle \mu, \hat{\mathbb{K}} \rangle$ is not a compact extension of $\mathbb{K}$. $\square$

Corollary 3.18. *The profinite limit and the canonical extension of a modal algebra are in general not the same.*

4. Introducing functors

In this Section we will introduce some category theory into our perspective on the profinite limit in an attempt to show that the profinite limit and the canonical extension of a modal algebra are related.

4.1. **Towards a categorical perspective.** We know from Theorem 3.13 that $\hat{\mathbb{A}}$ is perfect. This means that it is related to a Kripke frame $(\hat{\mathbb{A}})_+$, just like the canonical extension $\mathbb{A}^\sigma$ is related to the ultrafilter frame $\mathbb{A}_\bullet$. To show that $\mathbb{A}_\bullet$ and $(\hat{\mathbb{A}})_+$ are in fact related, we will bring in some more category theory.

Because partial orders are categories, we may see our diagram F as a functor from the category $\langle \Phi, \subseteq \rangle$ to $\mathbf{MA}$, the category of modal algebras with homomorphisms. We now call $\langle \Phi, \subseteq \rangle$ the index category of F. For a more extensive treatment of the category theory introduced below, see [1].

Definition 4.1. Let $G\colon \mathbf{I} \to \mathbf{C}$ be a diagram in some category $\mathbf{C}$ indexed by a small category $\mathbf{I}$. (A category is called small if it can be represented as a set.) A *cone* for the diagram $G\colon \mathbf{I} \to \mathbf{C}$ is a pair $\langle A, \big(\mu_X\colon A \to G(X)\big)_{X\in\mathbf{I}}\rangle$ such that for all $f\colon X \to Y$ in $\mathbf{I}$, the following diagram commutes in $\mathbf{C}$:

$$\begin{tikzcd}
A \arrow[d, "\mu_X"'] \arrow[dr, "\mu_Y"] & \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

A *map of cones* g from $\langle A, \mu\rangle$ to $\langle B, \nu\rangle$ is a $\mathbf{C}$-morphism $g\colon A \to B$ such that $\nu_X g = \mu_X$ for all X in $\mathbf{I}$. The *limit* of G, when it exists, is a cone $\langle \varprojlim G, \pi\rangle$ such that for every other cone for G $\langle A, \mu\rangle$ there is a unique map of cones $g\colon A \to \varprojlim G$:

$$\begin{tikzcd}
A \arrow[d, "\mu_X"] \arrow[dr, "\mu_Y"] \arrow[dd, dashed, bend right=60, "g"'] & \\
G(X) \arrow[r, "G(f)"'] & G(Y) \\
\varprojlim G \arrow[u, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}$$

Limits are unique up to isomorphism. Dually, $\langle A, (\mu_X\colon G(X) \to A)_{X\in\mathbf{I}}\rangle$ is a *cocone* for G if

$$\begin{tikzcd}
A & \\
G(X) \arrow[u, "\mu_X"] \arrow[r, "G(f)"'] & G(Y) \arrow[ul, "\mu_Y"']
\end{tikzcd}$$

commutes for every $f\colon X \to Y$ in $\mathbf{I}$. The *colimit* of G is a pair $\langle \varinjlim G, \iota\rangle$ such that for every cocone for G $\langle A, \mu\rangle$ there is a unique map of cocones $g\colon \varinjlim G \to A$.

$$\begin{tikzcd}
A & \\
G(X) \arrow[u, "\mu_X"'] \arrow[d, "\iota_X"] \arrow[r, "G(f)"'] & G(Y) \arrow[ul, "\mu_Y"'] \arrow[dl, "\iota_Y"] \\
\varinjlim G \arrow[uu, dashed, bend left=60, "g"] &
\end{tikzcd}$$

Cocones are also unique up to isomorphism. A category is *complete* (*cocomplete*) if it has a limit (colimit) for every diagram over a small index category.

In any category of algebras, a concrete example of $\varprojlim F$ is given by our construction in Definition 3.4. In categories of algebras, limits are subalgebras of products. Conversely, any product or subalgebra in a category of algebras can be construed as a limit with the right diagram.

In the category $\mathbf{KFr}$ of Kripke frames with bounded morphisms, we construct the colimit of a diagram $G\colon \mathbf{I} \to \mathbf{KFr}$ as follows: we take the disjoint union of all the objects in the diagram and form a quotient $\coprod_{\mathbf{I}} G(X)/{\sim}$, where for $x \in G(X)$ and $y \in G(Y)$ we have $x \sim y$ if there are $f_{XZ}\colon X \to Z$ and $f_{YZ}\colon Y \to Z$ such that $G(f_{XZ})(x) = G(f_{YZ})(y)$. If we assume that all the $G(f)\colon G(X) \hookrightarrow G(Y)$ are embeddings (which is quite natural as we will see below), this means that the colimit of a diagram of Kripke frames is a disjoint union of frames where two points $x \in G(X)$ and $y \in G(Y)$ in different frames are identified under $\sim$ if there is some bigger frame $G(Z)$ in the diagram such $G(X)$ and $G(Y)$ are both generated subframes of $G(Z)$ and x and y coincide in $G(Z)$.

So now we can speak of the cone $\langle \mathbb{A}, \mu \rangle$ for F and call $\hat{\mathbb{A}}$ the limit of F. How

$$\begin{tikzcd}
\mathbb{A} \arrow[rrd, two heads, "\mu_\theta"] \arrow[rdd, two heads, "\mu_{\theta\wedge\psi}"] \arrow[rrddd, two heads, bend right, "\mu_\psi"'] & & \\
& & \mathbb{A}/\theta \\
& \mathbb{A}/(\theta \wedge \psi) \arrow[ru, "\varphi_{(\theta\wedge\psi)\theta}"'] \arrow[rd, "\varphi_{(\theta\wedge\psi)\psi}"] & \\
& & \mathbb{A}/\psi
\end{tikzcd}$$

FIGURE 3. $\langle \mathbb{A}, \mu \rangle$ is a cone for F.

does this help us understand the dual of the profinite completion, $(\hat{\mathbb{A}})_+$?

4.2. **From cone to cocone.** We will take our diagram and cone and transport them to the category of Kripke frames **KFr**, after the following observations.

Because $(\cdot)_+ \colon \mathbf{MA}^+ \rightleftarrows \mathbf{KFr} \colon (\cdot)^+$ is a dual equivalence, $(\cdot)_+$ turns limits into colimits, and $(\cdot)^+$ turns colimits into limits. This quasi-preservation does not apply to $(\cdot)_\bullet$, however. The reason is that $(\cdot)_\bullet$ does not 'preserve' products, i.e. it does not turn products into disjoint unions: $\big(\prod_I \mathbb{A}_i\big)_\bullet \not\cong \coprod_I (\mathbb{A}_i)_\bullet$. In other words, the dual of the product is not the same as the coproduct (disjoint union) of the duals (this is a consequence of Theorem 1.9.10 of [10]).

Now recall that $\mu \colon \mathbb{A} \to \prod_\Phi \mathbb{A}/\theta$ is a homomorphism from $\mathbb{A}$ to a product. Therefore $\mu_\bullet \colon \big(\prod_\Phi \mathbb{A}/\theta\big)_\bullet \to \mathbb{A}_\bullet$ is a bounded morphism from the dual of the product to the dual of $\mathbb{A}$. This function is now different from $\big((\mu_\theta)_\bullet\big)_\Phi \colon \coprod_\Phi (\mathbb{A}/\theta)_\bullet \to \mathbb{A}_\bullet$, which is a bounded morphism from the coproduct of the duals to the dual of $\mathbb{A}$. Since it is the latter we care about, we will have to suffer some unpleasant notation.

Lemma 4.2. *The category* $\mathbf{MA}^+$ *is complete, whence* **KFr** *is cocomplete.*

Proof. Since the limit in a category of algebras can be constructed as in Definition 3.4, and we have shown in Section 3 that products and subalgebras of perfect modal algebras are perfect, the limit of any diagram in $\mathbf{MA}^+$ is also an object of $\mathbf{MA}^+$. □

Now we are ready to turn our cone of modal algebras into a cocone of Kripke frames as promised. By applying $(\cdot)_\bullet$ to all the arrows and objects of F we find a diagram of Kripke frames $F_\bullet$, see Figure 4 on the following page. Because $(\cdot)_\bullet$ is a contravariant functor, it follows from the fact that $\varphi_{\theta\theta'}\mu_\theta = \mu_{\theta'}$ for all $\theta, \theta' \in \Phi$ such that $\theta \subseteq \theta'$, that $(\mu_\theta)_\bullet(\varphi_{\theta\theta'})_\bullet = (\mu_{\theta'})_\bullet$, i.e. that $\langle \mathbb{A}_\bullet, \big((\mu_\theta)_\bullet\big)_\Phi \rangle$ is a cocone for $F_\bullet$ in **KFr**. So here we have a system of finite generated subframes of the ultrafilter frame of $\mathbb{A}$. In fact, these are *all* the finite generated subframes of $\mathbb{A}_\bullet$ (modulo isomorphism).

Lemma 4.3. *Let* $f \colon \mathfrak{F} \hookrightarrow \mathbb{A}_\bullet$ *be a bounded morphism and let* $\mathfrak{F}$ *be finite. Then there is some* $\theta \in \Phi$ *such that* $\mathfrak{F} \cong (\mathbb{A}/\theta)_\bullet$.

$$\begin{tikzcd}
& & \mathbb{A}_\bullet \\
(\mathbb{A}/\theta)_\bullet \arrow[urr, hook, "(\mu_\theta)_\bullet"] \arrow[dr, hook, "(\varphi_{(\theta\wedge\psi)\theta})_\bullet"'] & & \\
& (\mathbb{A}/(\theta\wedge\psi))_\bullet \arrow[uur, hook, "(\mu_{\theta\wedge\psi})_\bullet"'] & \\
(\mathbb{A}/\psi)_\bullet \arrow[ur, hook, "(\varphi_{(\theta\wedge\psi)\psi})_\bullet"] \arrow[uuurr, hook, bend right, "(\mu_\psi)_\bullet"'] & &
\end{tikzcd}$$

FIGURE 4. $\langle \mathbb{A}_\bullet, \big((\mu_\theta)_\bullet\big)_\Phi \rangle$ is a cocone for $F_\bullet$.

Proof. Recall that $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma = (\mathbb{A}_\bullet)^+$ is the canonical embedding. We will show that $f^+e\colon \mathbb{A} \twoheadrightarrow \mathfrak{F}^+$ is a surjection onto $\mathfrak{F}^+$, whence there must be a $\theta \in \Phi$ such that $\mathbb{A}/\theta \cong \mathfrak{F}^+$, so that $(\mathbb{A}/\theta)_\bullet \cong (\mathfrak{F}^+)_+ \cong \mathfrak{F}$.

Because $f\colon \mathfrak{F} \hookrightarrow \mathbb{A}_\bullet$ is an embedding, by duality we find that $f^+\colon (\mathbb{A}_\bullet)^+ = \mathbb{A}^\sigma \twoheadrightarrow \mathfrak{F}^+$ is a surjection. Let $y \in \mathfrak{F}^+$, then we want to find some element $a_y \in \mathbb{A}$ such that $f^+e(a_y) = y$ to show that $f^+e\colon \mathbb{A} \twoheadrightarrow \mathfrak{F}$ is a surjection. Since $f^+\colon \mathbb{A}^\sigma \twoheadrightarrow \mathfrak{F}^+$ is surjective, there must be some $x \in \mathbb{A}^\sigma$ such that $f^+(x) = y$. Because $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma$ is a dense embedding, we know that

$$x = \bigvee \big\{ \textstyle\bigwedge e[\nabla] \mid \bigwedge e[\nabla] \leq x \text{ and } \nabla \text{ is a filter of } \mathbb{A} \big\}.$$

Because f^+ is a complete homomorphism (thanks to duality), we find that

$$y = f^+(x) = f^+\Big(\bigvee \big\{ \textstyle\bigwedge e[\nabla] \mid \bigwedge e[\nabla] \leq x \big\} \Big) = \bigvee \big\{ \textstyle\bigwedge f^+ e[\nabla] \mid \bigwedge e[\nabla] \leq x \big\}.$$

Let

$$S_x = \big\{ \nabla \subseteq \mathbb{A} \mid \nabla \text{ is a filter of } \mathbb{A} \text{ such that } \textstyle\bigwedge e[\nabla] \leq x \big\}.$$

Since $\mathfrak{F}^+$ is finite, for $\nabla \in S_x$ we know that $\bigwedge f^+e[\nabla]$ is a finite meet, i.e. there must be $a_0, \dots a_n \in \nabla$ such that

$$\textstyle\bigwedge f^+ e[\nabla] = \bigwedge_{i \leq n} f^+ e(a_i) = f^+ e\big(\bigwedge_{i\leq n} a_i\big)$$

for some $n \in \omega$. Let $a_\nabla = \bigwedge_{i \leq n} a_i$, then $\bigwedge f^+e[\nabla] = f^+e(a_\nabla)$. Now

$$y = \bigvee \big\{ \textstyle\bigwedge f^+ e[\nabla] \mid \nabla \in S_x \big\} = \bigvee \{ f^+ e(a_\nabla) \mid \nabla \in S_x \}.$$

Again we use the fact that $\mathfrak{F}^+$ is finite to conclude that there must be $\nabla_0, \dots, \nabla_m \in S_x$ such that

$$y = \bigvee \{ f^+ e(a_\nabla) \mid \nabla \in S_x \} = \bigvee_{i \leq m} f^+ e(a_{\nabla_i}) = f^+ e\big(\textstyle\bigvee_{i \leq m} a_{\nabla_i}\big),$$

for some $m \in \omega$. We let $a_y = \bigvee_{i \leq m} a_{\nabla_i}$, so that $f^+e(a_y) = y$. Since y was arbitrary we conclude that $f^+e\colon \mathbb{A} \twoheadrightarrow \mathfrak{F}^+$ is a surjection. Since $\mathfrak{F}^+$ is finite there must be some $\theta \in \Phi$ such that $\mathbb{A}/\theta \cong \mathfrak{F}^+$, thus $\mathfrak{F} \cong (\mathbb{A}/\theta)_\bullet$. □

Remark 4.4. We introduced a duality for modal algebras in Section 2. There are in fact two dualities for modal algebras: a topological duality and a discrete duality (we have only introduced the latter). Using the topological duality theory for modal algebras, the lemma above becomes a well-known fact about the correspondence between closed generated subframes and homomorphic images of modal algebras.

Because **KFr** is cocomplete, there must be a colimit

$$\left\langle \varinjlim F_\bullet, \big(\iota_\theta \colon (\mathbb{A}/\theta)_\bullet \to \varinjlim F_\bullet\big)_\Phi \right\rangle.$$

But then there must also be a map of cocones $h\colon \varinjlim F_\bullet \to \mathbb{A}_\bullet$, see Figure 5.

$$\begin{tikzcd}
\mathbb{A}_\bullet & \\
& \boxed{F_\bullet(\Phi)} \arrow[ul, "(\mu_\theta)_\bullet"'] \arrow[d, "\iota_\theta"] \\
& \varinjlim F_\bullet \arrow[uul, dashed, "h"]
\end{tikzcd}$$

FIGURE 5. The bounded morphism h is a map of cocones.

Lemma 4.5. *The bounded morphism $h\colon \varinjlim F_\bullet \to \mathbb{A}_\bullet$ is injective.*

Proof. Suppose that $x/{\sim}, y/{\sim} \in \varinjlim F_\bullet$ and $x/{\sim} \neq y/{\sim}$. We want to show that $h(x/{\sim}) \neq h(y/{\sim})$. In order to do that, we want to go from $\varinjlim F_\bullet$ to an object of $F_\bullet$. However, we only know that $x \in (\mathbb{A}/\theta)_\bullet$ and $y \in (\mathbb{A}/\psi)_\bullet$ and those two frames need not be the same generated subframe of $\mathbb{A}_\bullet$. Therefore we first show that without loss of generality, we may assume that x and y are points in the same frame of $F_\bullet$. Suppose not, then we want to compare x and y in a bigger frame that contains $(\mathbb{A}/\theta)_\bullet$ and $(\mathbb{A}/\psi)_\bullet$ as generated subframes. What we know so far is that $x \in (\mathbb{A}/\theta)_\bullet$, $y \in (\mathbb{A}/\psi)_\bullet$ and $\iota_\theta(x) = x/{\sim}$ and $\iota_\psi(y) = y/{\sim}$. Then because

$$\begin{tikzcd}
& \mathbb{A}/\theta \\
\mathbb{A}/(\theta \wedge \psi) \arrow[ur, two heads, "\varphi_{(\theta\wedge\psi)\theta}"] \arrow[dr, two heads, "\varphi_{(\theta\wedge\psi)\psi}"] & \\
& \mathbb{A}/\psi
\end{tikzcd}$$

we find that

$$\begin{tikzcd}
(\mathbb{A}/\theta)_\bullet \arrow[dr, hook, "(\varphi_{(\theta\wedge\psi)\theta})_\bullet"] & \\
& \big(\mathbb{A}/(\theta \wedge \psi)\big)_\bullet \\
(\mathbb{A}/\psi)_\bullet \arrow[ur, hook, "(\varphi_{(\theta\wedge\psi)\psi})_\bullet"'] &
\end{tikzcd}$$

Set $x' = (\varphi_{(\theta\wedge\psi)\theta)})_\bullet(x)$ and $y' = (\varphi_{(\theta\wedge\psi)\psi})_\bullet(y)$, then x' and y' are essentially the same points as x and y but now they live in the same object of $F_\bullet$, viz. $\big(\mathbb{A}/(\theta\wedge\psi)\big)_\bullet$. We also have $\iota_{\theta\wedge\psi}(x') = x/{\sim}$ and $\iota_{\theta\wedge\psi}(y') = y/{\sim}$, because $(\iota_\theta)_{\theta\in\Phi}$ is a cocone for

$F_\bullet$, so e.g. $\iota_{\theta\wedge\psi}\varphi_{(\theta\wedge\psi)\theta} = \iota_\theta$:

$$\begin{tikzcd}
(\mathbb{A}/\theta)_\bullet \arrow[r, hook, "\varphi_{(\theta\wedge\psi)\theta}"] \arrow[d, hook, "\iota_\theta"'] & \big(\mathbb{A}/(\theta\wedge\psi)\big)_\bullet \arrow[dl, hook, "\iota_{\theta\wedge\psi}"] \\
\varinjlim F_\bullet &
\end{tikzcd}$$

Now it must be the case that $x' \neq y'$, for to assume otherwise would imply that $x/{\sim} = y/{\sim}$, contrary to our initial assumption. We conclude that to two different elements $x/{\sim}, y/{\sim} \in \varinjlim F_\bullet$ there must correspond two different elements x', y' in some object of $F_\bullet$.

Beginning again: Let $x/{\sim}, y/{\sim} \in \varinjlim F_\bullet$ with $x/{\sim} \neq y/{\sim}$, then we now know that we can assume that $x, y \in (\mathbb{A}/\theta)_\bullet$ (so $\iota_\theta(x) = x/{\sim}$ and $\iota_\theta(y) = y/{\sim}$) for some $\theta \in \Phi$, and that $x \neq y$. Because $\mu_\theta\colon \mathbb{A} \twoheadrightarrow \mathbb{A}/\theta$ is a surjection, $(\mu_\theta)_\bullet\colon (\mathbb{A}/\theta)_\bullet \hookrightarrow \mathbb{A}_\bullet$ is an embedding. Therefore $(\mu_\theta)_\bullet(x) \neq (\mu_\theta)_\bullet(y)$ in $\mathbb{A}_\bullet$. Now we use the fact that $h\colon \varinjlim F_\bullet \to \mathbb{A}_\bullet$ is a map of cocones:

$$\begin{tikzcd}
& \mathbb{A}_\bullet \\
(\mathbb{A}/\theta)_\bullet \arrow[ur, hook, "(\mu_\theta)_\bullet"] \arrow[d, hook, "\iota_\theta"] & \\
\varinjlim F_\bullet \arrow[uur, bend right, "h"'] &
\end{tikzcd}$$

i.e. $h\iota_\theta = (\mu_\theta)_\bullet$. Since $(\mu_\theta)_\bullet(x) \neq (\mu_\theta)_\bullet(y)$, it follows that $h\iota_\theta(x) \neq h\iota_\theta(y)$. We know that $\iota_\theta(x) = x/{\sim}$ and $\iota_\theta(y) = y/{\sim}$, so this means that $h(x/{\sim}) \neq h(y/{\sim})$. We conclude that h is injective, since x and y were arbitrary. □

4.3. **From colimit to limit.** We may now take our diagram and its cocones back to the algebra side using $(\cdot)^+$, where we find that $h^+\colon \mathbb{A}^\sigma \twoheadrightarrow \big(\varinjlim F_\bullet\big)^+$, see Figure 6. Note that h^+ is a complete homomorphism, because $(\cdot)^+$ creates complete homomorphisms. Because finite modal algebras are isomorphic to their canoni-

$$\begin{tikzcd}
& \mathbb{A}^\sigma \arrow[dl, shift right, "(\mu_\theta)^\sigma"'] \arrow[dl, shift left] \arrow[ddl, bend left, "h^+"] \\
\boxed{F^\sigma(\Phi)} & \\
\big(\varinjlim F_\bullet\big)^+ \arrow[u, shift left, "(\iota_\theta)^+"] \arrow[u, shift right] &
\end{tikzcd}$$

FIGURE 6. Two cones for our new diagram F^σ.

cal extensions, $\langle \mathbb{A}^\sigma, \big((\mu_\theta)^\sigma\big)_\Phi \rangle$ and $\langle \big(\varinjlim F_\bullet\big)^+, \iota^+ \rangle$ are also cones for F (modulo isomorphism).

So the homomorphisms $(\mu_\theta)^\sigma$ take us from $\mathbb{A}^\sigma$ to $\mathbb{A}/\theta$ modulo isomorphism for all $\theta \in \Phi$; we would like to find maps $\nu_\theta\colon \mathbb{A}^\sigma \twoheadrightarrow \mathbb{A}/\theta$ that get us there directly.

To that end we will have to make explicit the isomorphisms that we would rather ignore and do some rewriting.

The elements of $\mathbb{A}^\sigma$ are sets of ultrafilters of $\mathbb{A}$; to understand what ν does, we must first understand what $(\mu_\theta)_\bullet\colon (\mathbb{A}/\theta)_\bullet \hookrightarrow \mathbb{A}_\bullet$ does for $\theta \in \Phi$. Note that all filters in a finite Boolean algebra are principal and that principal ultrafilters are generated by atoms. Therefore the elements of $(\mathbb{A}/\theta)_\bullet$ are principal filters $(a/\theta)\!\uparrow$ where a/θ is an atom of $\mathbb{A}/\theta$. Knowing this we can unravel what $(\mu_\theta)_\bullet\colon (\mathbb{A}/\theta)_\bullet \hookrightarrow \mathbb{A}_\bullet$ does:

$$(\mu_\theta)_\bullet\big((a/\theta)\!\uparrow\big) = \mu_\theta^{-1}\big((a/\theta)\!\uparrow\big) = \{b \in \mathbb{A} \mid a/\theta \leq b/\theta\}.$$

Note that duality tells us that the right-hand side is (apparently) an ultrafilter of $\mathbb{A}$. Now $(\mu_\theta)^\sigma = ((\mu_\theta)_\bullet)^+\colon \mathbb{A}^\sigma \twoheadrightarrow (\mathbb{A}/\theta)^\sigma$ almost takes us from $\mathbb{A}^\sigma$ to $\mathbb{A}/\theta$. Let $X \in \mathbb{A}^\sigma$ be a set of ultrafilters of $\mathbb{A}$. Then

$$(\mu_\theta)^\sigma(X) = ((\mu_\theta)_\bullet)^+(X) = (\mu_\theta)_\bullet^{-1}(X) = \{(a/\theta)\!\uparrow \mid (\mu_\theta)_\bullet\big((a/\theta)\!\uparrow\big) \in X\}.$$

When we substitute what $(\mu_\theta)_\bullet$ does in the right-hand side above, we find that

$$(\mu_\theta)^\sigma(X) = \big\{(a/\theta)\!\uparrow \mid \{b \mid a/\theta \leq b/\theta\} \in X\big\}.$$

This tells us which set of ultrafilters of $\mathbb{A}/\theta$ goes with each element of $\mathbb{A}^\sigma$. Now we would like to know how to get to $\mathbb{A}/\theta$, not $(\mathbb{A}/\theta)^\sigma$. For that we use the fact that the isomorphism from $(\mathbb{A}/\theta)^\sigma$ to $\mathbb{A}/\theta$ is the following:

$$\varepsilon\colon X \mapsto \bigvee_{\nabla \in X} \bigwedge \nabla.$$

So if $\nabla = (a/\theta)\!\uparrow$ then $\varepsilon(\{\nabla\}) = \bigwedge (a/\theta)\!\uparrow = a/\theta$. We find that to get from $\mathbb{A}^\sigma$ to $\mathbb{A}/\theta$ we need the following:

$$\nu_\theta\colon X \mapsto \bigvee \big\{a/\theta \in (\mathbb{A}/\theta)_+ \mid \{b \mid a/\theta \leq b/\theta\} \in X\big\},$$

where we remind the reader that the elements of $(\mathbb{A}/\theta)_+$ are the atoms of $\mathbb{A}/\theta$.

Now what about this connection between $(\hat{\mathbb{A}})_+$ and $\mathbb{A}_\bullet$ we promised? Because $(\cdot)^+$ preserves limits, we find that $\big(\varinjlim F_\bullet\big)^+ \cong \varprojlim (F_\bullet)^+ = \varprojlim F^\sigma$. Because F and F^σ are the same diagrams modulo isomorphism, it is the case that $\varprojlim F \cong \varprojlim F^\sigma$. This means that $(\hat{\mathbb{A}})_+ \cong \varinjlim F_\bullet$. Now we see two things: that $(\hat{\mathbb{A}})_+$ is a generated subframe of the ultrafilter frame $\mathbb{A}_\bullet$ and that (equivalently) $\hat{\mathbb{A}}$ is a homomorphic image of $(\mathbb{A}_\bullet)^+ = \mathbb{A}^\sigma$, the canonical extension of $\mathbb{A}$.

We pause for a moment to review what we have found thus far. Given a modal algebra $\mathbb{A}$, we constructed a diagram of finite quotients F, in which quotients are homomorphic images of each other. Dually, this corresponds to a diagram $F_\bullet$ of finite generated subframes of the ultrafilter frame $\mathbb{A}_\bullet$ of $\mathbb{A}$, where the finite generated subframes in $F_\bullet$ are generated subframes of each other. Now the colimit of this diagram $F_\bullet$ is the generated subframe of the ultrafilter frame of $\mathbb{A}$ that contains every finite generated subframe of $\mathbb{A}_\bullet$ as a generated subframe of itself (it is the hereditarily finite subframe of $\mathbb{A}_\bullet$). The limit of F on the other hand, is a homomorphic image of $\mathbb{A}^\sigma$ which has every finite quotient of $\mathbb{A}$ as a quotient of itself.

We continue our exposition with a summarizing theorem.

Theorem 4.6. *Let $\mathbb{A}$ be a modal algebra. Let $\nu\colon \mathbb{A}^\sigma \to \hat{\mathbb{A}}$ be defined by*

$$\nu_\theta(X) = \bigvee \big\{a/\theta \in (\mathbb{A}/\theta)_+ \mid \{b \mid a/\theta \leq b/\theta\} \in X\big\}.$$

Then $\nu\colon \mathbb{A}^\sigma \twoheadrightarrow \hat{\mathbb{A}}$ is a complete, surjective homomorphism.

We can improve on this. The canonical extension is a completion, but we have not introduced the embedding witnessing this yet.

Definition 4.7. We define $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma$ to be the Stone embedding, i.e.

$$e\colon a \mapsto \{\nabla \in \mathbb{A}_\bullet \mid a \in \nabla\}.$$

Now we have three connections: $\mu\colon \mathbb{A} \to \hat{\mathbb{A}}$, $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma$ and $\nu\colon \mathbb{A}^\sigma \twoheadrightarrow \hat{\mathbb{A}}$. How do these connections relate?

Lemma 4.8. *$e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma$ is a map of cones.*

Proof. Let $\theta \in \Phi$, then we must show that $\nu_\theta e = \mu_\theta$. Let $a \in \mathbb{A}$, then

$$\nu_\theta e(a) = \bigvee \big\{ b/\theta \in (\mathbb{A}/\theta)_+ \mid \{c \mid b/\theta \leq c/\theta\} \in e(a) \big\}$$
$$= \bigvee \big\{ b/\theta \mid a \in \{c \mid b/\theta \leq c/\theta\} \big\} = \bigvee \{ b/\theta \in (\mathbb{A}/\theta)_+ \mid b/\theta \leq a/\theta \}.$$

Because $\mathbb{A}/\theta$ is complete and atomic, any element of $\mathbb{A}/\theta$ is equal to the join of the atoms below it. But then it follows that

$$\nu_\theta e(a) = a/\theta = \mu_\theta(a).$$

Because a was arbitrary, it follows that $\nu_\theta e = \mu_\theta$. We conclude that e is a map of cones. $\square$

The following theorem is a direct consequence of the previous lemma:

Theorem 4.9. *Let $\mathbb{A}$ be a modal algebra. Let $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma$ be the Stone embedding, let $\nu\colon \mathbb{A}^\sigma \twoheadrightarrow \hat{\mathbb{A}}$ be the surjection of Theorem 4.6 and let $\mu\colon \mathbb{A} \to \hat{\mathbb{A}}$ be the natural map to the profinite limit. Then $\nu e = \mu$, i.e. the homomorphism μ from $\mathbb{A}$ to $\hat{\mathbb{A}}$ can be extended to a surjection ν from $\mathbb{A}^\sigma$ to $\hat{\mathbb{A}}$.*

We can now draw the picture shown in Figure 7, showing the relation between a modal algebra, its canonical extension and its profinite limit. To conclude this section, we show one consequence of the previous theorem.

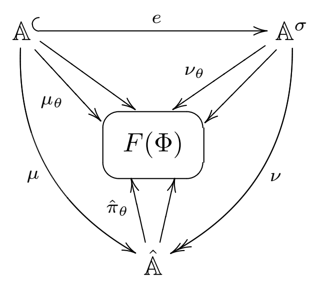

FIGURE 7. The surjection ν is an extension of μ via $\mathbb{A}^\sigma$.

Lemma 4.10. *Assume that $\mathbb{A}$ is residually finite. Then $\mu\colon \mathbb{A} \hookrightarrow \hat{\mathbb{A}}$ is a dense embedding.*

Proof. Let $\alpha \in \hat{\mathbb{A}}$, then there is $X \in \mathbb{A}^\sigma$ such that $\nu(X) = \alpha$. Because $\mathbb{A}$ is dense in $\mathbb{A}^\sigma$,

$$X = \bigvee \big\{ \bigwedge e[\nabla] \mid \bigwedge e[\nabla] \leq X,\ \nabla \text{ is a filter of } \mathbb{A} \big\},$$

whence

$$\nu(X) = \nu\Big(\bigvee \big\{ \bigwedge e[\nabla] \mid \bigwedge e[\nabla] \leq X \big\} \Big) = \bigvee \big\{ \bigwedge \nu e[\nabla] \mid \bigwedge e[\nabla] \leq X \big\},$$

using the fact that ν is a complete homomorphism twice. By the previous theorem, we know that $\nu e = \mu$, whence

$$\alpha = \nu(X) = \bigvee \big\{ \bigwedge \mu[\nabla] \mid \bigwedge e[\nabla] \leq X \big\},$$

so α is a join of closed elements. One may prove that α is a meet of opens in a similar fashion; we therefore conclude that $\mu\colon \mathbb{A} \hookrightarrow \hat{\mathbb{A}}$ is a dense embedding. $\square$

In [11] it is shown that $\mu[\mathbb{A}]$ is a dense subalgebra of $\hat{\mathbb{A}}$. There the result is used to show that the profinite limit and the canonical extension are equal (assuming $\mathbb{A}$ is finitely generated), while here we use the fact that the profinite limit and the canonical extension are related (though not always equal) to prove the density result.

Remark 4.11. Only Lemma 4.10 above requires that $\mathbb{A}$ be residually finite, and note that even in that proof nowhere do we use the fact that μ is an embedding.

5. Generalizing the functorial approach

In this Section we will argue that the techniques of Section 4 are more widely applicable than just in the category of modal algebras. We will go through our proofs of Theorems 4.6 and 4.9 again.

In the proofs of the theorems we mostly use a combination of universal algebra and duality via functors. We will show that this can easily be generalized. Moreover we claim that when we do look at the internal structure of our algebra, such as in the rewriting prior to Theorem 4.6 and in Lemma 4.8, what we do is in reality still fairly general.

The essence of the proofs of our theorems in Section 4 is the use of duality via functors. There are a number of categories of algebras which have a similar duality theory. In this section, when we say algebra we think of distributive lattices, Heyting algebras, Boolean algebras, distributive lattices with operators, Heyting algebras with operators or Boolean algebras with operators. In each case there are two dualities: a topological duality for the entire category of algebras, and a discrete duality for the subcategory of 'perfect' algebras. In the case of modal algebras, we have a topological duality between arbitrary modal algebras and topological Kripke frames (also known as descriptive general frames), and a discrete duality between perfect modal algebras and 'ordinary' Kripke frames. For a treatment of the duality theory of distributive lattices with operators, the reader may consult [9].

The canonical extension of an algebra is created by taking its topological dual, forgetting the topology and taking the perfect algebra corresponding to the resulting discrete structure, see Figure 8. The ultrafilter frame $\mathbb{A}_\bullet$ of a modal algebra

$$\begin{tikzcd}
\text{algebra} \arrow[r, "\substack{\text{topological}\\\text{duality}}"] \arrow[d, "\substack{\text{canonical}\\\text{extension}}"'] & \substack{\text{structured}\\\text{topology}} \arrow[d, "\substack{\text{forget}\\\text{topology}}"] \\
\substack{\text{perfect}\\\text{algebra}} & \substack{\text{discrete}\\\text{structure}} \arrow[l, "\substack{\text{discrete}\\\text{duality}}"']
\end{tikzcd}$$

Figure 8. Constructing the canonical extension using duality.

$\mathbb{A}$ that we introduced before is really the result of taking the topological dual of $\mathbb{A}$ and forgetting its topology. Therefore we here again call the functor combining 'topological duality' and 'forgetting topology' $(\cdot)_\bullet$. We call the functors witnessing discrete duality $(\cdot)_+$ (which previously created the atom structure of a modal algebra) and $(\cdot)^+$ (which creates the complex algebra of a discrete structure). As a reminder we provide the picture of the duality for modal algebras in Figure 9. The

$$\begin{tikzcd}
\text{modal algebra} \arrow[r, "\text{topological duality}"] \arrow[d, "(\cdot)^\sigma"'] \arrow[dr, "(\cdot)_\bullet"] & \text{topological Kripke frame} \arrow[d, "\text{forget topology}"] \\
\text{perfect modal algebra} & \text{Kripke frame} \arrow[l, "(\cdot)^+"']
\end{tikzcd}$$

FIGURE 9. Constructing the canonical extension of a modal algebra.

dualities of all the above mentioned algebras have in common that if we construct $\mathbb{A}^\sigma$ using duality, the elements of $\mathbb{A}^\sigma$ are (special) sets of prime filters. Recall that when we were talking about modal algebras, the elements of the canonical extension were sets of ultrafilters, and since ultrafilters in modal algebras are prime filters and vice versa, we might as well have described the duality for modal algebras using prime filters.

So let us consider our approach of Section 4 even more abstractly. Let $\mathbb{A}$ be an algebra, then we start by considering $\mathbb{A}$ as a cone seeing all its finite quotients again (see Figure 10). Here we only use universal algebra. Because each of the

$$\begin{tikzcd}
\mathbb{A} \arrow[dr, shift right, "\mu_\theta"'] \arrow[dr, shift left] & \\
& \boxed{F(\Phi)}
\end{tikzcd}$$

FIGURE 10. $\mathbb{A}$ is a cone for its diagram of finite quotients F.

categories under consideration has its own version of Lemma 2.9, i.e. surjections become embeddings under $(\cdot)_\bullet$, we get Figure 11 again in the category of discrete structures where the colimit of the diagram of discrete duals of finite quotients of $\mathbb{A}$ is embedded in $\mathbb{A}_\bullet$, the 'prime filter frame' of $\mathbb{A}$ for lack of a better name. This is a combination of duality and the fact that our category of perfect algebras is complete (has limits), thus the category of discrete structures is cocomplete (has colimits). Going back to perfect algebras using discrete duality, we find that the canonical

$$\begin{tikzcd}
\mathbb{A}_\bullet & \\
& \boxed{F_\bullet(\Phi)} \arrow[ul, shift left, "(\mu_\theta)_\bullet"'] \arrow[ul, shift right] \arrow[d, shift right] \arrow[d, shift left, "\iota_\theta"] \\
& \varinjlim F_\bullet \arrow[uul, hook, bend left, "h"]
\end{tikzcd}$$

FIGURE 11. The colimit of $F_\bullet$ can be embedded in $\mathbb{A}_\bullet$.

extension $\mathbb{A}^\sigma = (\mathbb{A}_\bullet)^+$ of $\mathbb{A}$ is a cone for the diagram F^σ of canonical extensions of finite quotients of $\mathbb{A}$, see Figure 12, and that $\varprojlim F^\sigma$ is a homomorphic image of $\mathbb{A}^\sigma$. This is an application of duality, where we use the fact that embeddings become surjections under $(\cdot)^+$ and the fact that $(\cdot)^+$ turns colimits into limits. Now we use

$$\begin{tikzcd}
 & \mathbb{A}^\sigma \arrow[dl] \arrow[dl, shift left] \arrow[ddl, bend left, two heads] \\
\boxed{F^\sigma(\Phi)} & \\
\left(\varinjlim F_\bullet\right)^+ \arrow[u] \arrow[u, shift right] &
\end{tikzcd}$$

FIGURE 12. $\mathbb{A}^\sigma$ is a cone F_σ.

the fact that the canonical extension $(\mathbb{A}/\theta)^\sigma$ of a finite algebra $\mathbb{A}/\theta$ is isomorphic to $\mathbb{A}/\theta$ itself, whence we find that $\mathbb{A}^\sigma$ is a cone for F using some family of maps $(\nu_\theta\colon \mathbb{A}^\sigma \twoheadrightarrow \mathbb{A}/\theta)_{\theta\in\Phi}$. Without going into the details we can report that the maps $\nu_\theta\colon \mathbb{A}^\sigma \twoheadrightarrow \mathbb{A}/\theta$ are the same as before, i.e.

$$\nu_\theta(X) = \bigvee \left\{a/\theta \in (\mathbb{A}/\theta)_+ \mid \{b \mid a/\theta \le b/\theta\} \in X\right\},$$

where it should be noted that the elements of $(\mathbb{A}/\theta)_+$ are no longer atoms but *completely join prime* elements of $\mathbb{A}/\theta$, i.e. elements $a/\theta \in \mathbb{A}/\theta$ such that for all $\{b_i/\theta \mid i \in I\} \subseteq \mathbb{A}/\theta$ with $a/\theta \le \bigvee_I b_i/\theta$, there must be some $j \in I$ such that $a/\theta \le b_j/\theta$.

Now we have arrived at the situation of Theorem 4.6, which we sketch again in Figure 13. What remains to be studied is the canonical embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma$.

$$\begin{tikzcd}
\mathbb{A} \arrow[dr, "\mu_\theta"'] \arrow[dr, shift left] \arrow[ddr, bend right, "\mu"'] & & \mathbb{A}^\sigma \arrow[dl, "\nu_\theta"'] \arrow[dl, shift left] \arrow[ddl, bend left, two heads, "\nu"] \\
 & \boxed{F(\Phi)} & \\
 & \hat{\mathbb{A}} \arrow[u, "\hat{\pi}_\theta"] \arrow[u, shift right] &
\end{tikzcd}$$

FIGURE 13. $\langle \mathbb{A}^\sigma, \nu\rangle$ is a cone for F and $\hat{\mathbb{A}}$ is a homomorphic image of $\mathbb{A}^\sigma$.

Since the definition does not change, i.e. we still have

$$e\colon a \mapsto \{\nabla \in \mathbb{A}_\bullet \mid a \in \nabla\},$$

we get a new version of Lemma 4.8 saying that for all $\theta \in \Phi$, $\nu_\theta e = \mu_\theta$, i.e. we get the picture of Figure 14.

We now present a theorem in the absence of rigorous proof:

Theorem 5.1. *Let $\mathbb{A}$ be a distributive lattice, a Heyting algebra, a Boolean algebra, a distributive lattice with operators, a Heyting algebra with operators or a Boolean*

$$
\begin{tikzcd}
\mathbb{A} \arrow[rr, hook, "e"] \arrow[dr, shift left] \arrow[dr, shift right, "\mu_\theta"'] \arrow[ddr, bend right, "\mu"'] & & \mathbb{A}^\sigma \arrow[dl, shift left] \arrow[dl, shift right, "\nu_\theta"'] \arrow[ddl, bend left, "\nu"] \\
& \boxed{F(\Phi)} & \\
& \hat{\mathbb{A}} \arrow[u, shift left] \arrow[u, shift right, "\hat{\pi}_\theta"] &
\end{tikzcd}
$$

FIGURE 14. The map $\mu\colon \mathbb{A} \to \hat{\mathbb{A}}$ is extended to a surjection $\nu\colon \mathbb{A}^\sigma \twoheadrightarrow \hat{\mathbb{A}}$.

algebra with operators. Let $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma$ be the canonical embedding. For all $\theta \in \Phi$, define $\mu_\theta\colon \mathbb{A} \twoheadrightarrow \mathbb{A}/\theta$ by

$$\mu_\theta\colon a \mapsto a/\theta$$

and $\nu_\theta\colon \mathbb{A}^\sigma \twoheadrightarrow \mathbb{A}/\theta$ by

$$\nu_\theta(X) = \bigvee \big\{a/\theta \in (\mathbb{A}/\theta)_+ \mid \{b \mid a/\theta \leq b/\theta\} \in X\big\}.$$

Then $\nu\colon \mathbb{A}^\sigma \twoheadrightarrow \hat{\mathbb{A}}$ is a surjection and $\nu e = \mu$, i.e. the homomorphism μ from $\mathbb{A}$ to $\hat{\mathbb{A}}$ can be extended to a surjection ν from $\mathbb{A}^\sigma$ to $\hat{\mathbb{A}}$.

6. Beyond duality

We now have a result about distributive lattices with operators, Theorem 5.1, which we have phrased and proved without ever mentioning the operators. We have only spoken about joins, meets, join prime elements and order. In other words, we only looked at the distributive lattice underlying $\mathbb{A}$. There are two reasons why we were able to prove a result about lattice expansions while only looking at lattices. Firstly, we used a duality theory that does a lot of work for us. Secondly, the canonical embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma$ is definable using lattice properties only (recall that we defined it as a dense, compact completion of $\mathbb{A}$ in Definition 2.12). These two circumstances account for the fact that we could prove facts about lattice expansions while only worrying about the lattices. However, the more general the categories become (from Boolean algebras with operators to arbitrary lattice expansions) the more complicated is the duality theory. Therefore we will try to make less use of duality theory in this section to prove a more general version of Theorem 5.1.

So far we have always considered a lattice expansion (in Sections 3, 4 a modal algebra) as a whole. We can also choose to distinguish the lattice and the added operations, which is in fact the more common approach when studying completions of lattice expansions. We will strategically alternate between the whole lattice expansion $\mathbb{A} = \langle A, \vee, \wedge, 0, 1, \diamond\rangle$ and its *lattice skeleton* $\mathbb{A}_L = \langle A, \vee, \wedge, 0, 1\rangle$ where we consider $\diamond\colon \mathbb{A}_L \to \mathbb{A}_L$ as a separate function. Because we will be dealing with both lattice homomorphisms and lattice expansion homomorphisms, we introduce the abbreviations *DL-homomorphism* and *DLE-homomorphism* for the sake of brevity. Because the operations of lattice expansion are in general not smooth, we will have to consider two canonical extensions, $\mathbb{A}^\sigma$ and $\mathbb{A}^\pi$.

6.1. **Lattices only.** Below we will try to prove Theorem 5.1 for the lattice skeletons of $\mathbb{A}$, $\hat{\mathbb{A}}$ and $\mathbb{A}^\sigma$ only. The trick we use is forgetting the operations temporarily, so we can let duality for distributive lattices (without operations) do our work for us.

This way we can show that the lattice skeleton of $\hat{\mathbb{A}}$ is a homomorphic image of the lattice skeleton of $\mathbb{A}^\sigma$.

Definition 6.1. Let **DLE** be the category of bounded distributive lattice expansions with a unary operation[2] and DLE-homomorphisms and let **DL** be the category of bounded distributive lattices with DL-homomorphisms. We define the functor $(\cdot)_L \colon \mathbf{DLE} \to \mathbf{DL}$ which maps a lattice expansion $\mathbb{A}$ to its lattice skeleton $\mathbb{A}_L$ and a DLE-homomorphism $f \colon \mathbb{A} \to \mathbb{B}$ to itself: $f_L = f \colon \mathbb{A}_L \to \mathbb{B}_L$. In category theory $(\cdot)_L$ is called a *forgetful functor*.

Take a lattice expansion $\mathbb{A}$ and consider the cone $\langle \mathbb{A}, \mu \rangle$ for its diagram of finite quotients F. It follows that $\langle \mathbb{A}_L, \mu \rangle$ is a cone for F_L, the diagram of lattice skeletons of finite quotients of $\mathbb{A}$. Because we have a functor $(\cdot)^\sigma$ to create canonical extensions in the category of distributive lattices **DL**, we get the situation depicted in Figure 15, similar to what we found before in Section 5. Note that F_L is not the diagram associated with the profinite completion of $\mathbb{A}_L$.

$$
\begin{tikzcd}
\mathbb{A}_L \arrow[rr, hook, "e"] \arrow[dr, "\mu_\theta"] \arrow[ddr, bend right, "\mu"'] & & (\mathbb{A}_L)^\sigma \arrow[dl, "\nu_\theta"'] \arrow[ddl, bend left, "\nu"] \\
& F_L(\Phi) & \\
& \varprojlim F_L \arrow[u, "\hat{\pi}_\theta"] &
\end{tikzcd}
$$

FIGURE 15. The lattice skeleton of $\mathbb{A}$ can either be mapped to the limit of F_L directly using μ or via the canonical extension of $\mathbb{A}_L$ using νe.

Now we use a crucial insight:

Lemma 6.2. *The forgetful functor $(\cdot)_L$ preserves limits.*

But this means that the lattice skeleton of the profinite limit $\hat{\mathbb{A}}$ is isomorphic to the lattice $\varprojlim F_L$ introduced above, as $(\hat{\mathbb{A}})_L = (\varprojlim F)_L = \varprojlim F_L$ by the lemma above. Since $\varprojlim F_L$ is a homomorphic image of $(\mathbb{A}_L)^\sigma$, the canonical extension of the skeleton of $\mathbb{A}$, and $\varprojlim F_L$ is the lattice skeleton of $\hat{\mathbb{A}}$, it follows that the lattice skeleton of $\hat{\mathbb{A}}$ is a homomorphic image of the canonical extension of the lattice skeleton of $\mathbb{A}$. Note that we are not saying that $(\hat{\mathbb{A}})_L$ and $\widehat{(\mathbb{A}_L)}$ are the same, unlike the situation with the canonical extension, where $(\mathbb{A}_L)^\sigma = (\mathbb{A}^\sigma)_L = (\mathbb{A}^\pi)_L$.[3]

Theorem 6.3. *Let $\mathbb{A}$ be a distributive lattice expansion and let $\mathbb{A}_L$ be its lattice skeleton. Then the lattice skeleton of the profinite limit $(\hat{\mathbb{A}})_L$ is a homomorphic image of $(\mathbb{A}_L)^\sigma$.*

We will try to sketch this picture in Figure 16, where we write $f \colon \mathbb{B} \to_L \mathbb{C}$ to indicate that $f \colon \mathbb{B}_L \to \mathbb{C}_L$ is only a DL-homomorphism, not a DLE-homomorphism.

[2]For the sake of brevity we restrict ourselves to a single unary added operation again.

[3]See Section 7 for a discussion of this subject.

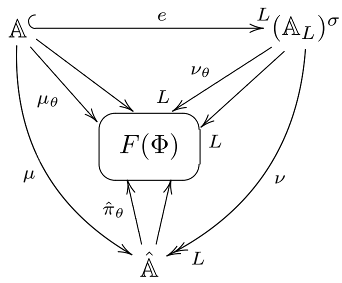

FIGURE 16. In this diagram, not all arrows are lattice *expansion* homomorphisms.

6.2. **No more duality.** Up until now we have relied greatly on duality. However, we will now forget about duality and remember the algebraic definition of the canonical extension in Definitions 2.12 and 2.13. We now think of $(\cdot)^\sigma$ (or $(\cdot)^\pi$) as a way of extending lattice expansions, and of extending functions between lattice expansions. The reason we do not speak of functors anymore is that there are DLE-homomorphisms $f\colon \mathbb{B} \to \mathbb{C}$ such that their extension $f^\sigma\colon \mathbb{B}^\sigma \to \mathbb{C}^\sigma$ is not a DLE-homomorphism (see Example 3.8 of [8]). This is bad news. Therefore we emphasize that for the duration of this section, $(\cdot)^\sigma$ is no longer a functor, but a means of extending lattice expansions and functions between lattice expansions.

There is also good news however, and that is that if $f\colon \mathbb{B} \twoheadrightarrow \mathbb{C}$ is a *surjective* DLE-homomorphism, then f is smooth (so $f^\sigma = f^\pi$) and $f^\sigma\colon \mathbb{B}^\sigma \twoheadrightarrow \mathbb{C}^\sigma$ is also a surjective DLE-homomorphism (see Corollary 2.28 and Theorem 3.7 of [8]).

We would now like to dress up the lattice skeletons again and still preserve the situation of Figure 16. To dress up $(\mathbb{A}_L)^\sigma$ we must choose the lower or the upper extension ($\mathbb{A}^\sigma$ or $\mathbb{A}^\pi$). We then get a picture like Figure 17 (note that for our diagram it does not matter if we pick $\mathbb{A}^\sigma$ or $\mathbb{A}^\pi$, because the ν_θ are defined on the skeleton, and $(\mathbb{A}^\sigma)_L = (\mathbb{A}^\pi)_L$).

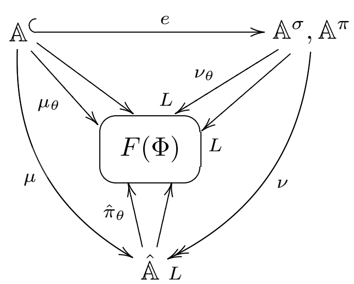

FIGURE 17. Because ν is only a DL-homomorphism, it does not matter if we pick $\mathbb{A}^\sigma$ or $\mathbb{A}^\pi$.

Now we recall from Section 4 that $\nu_\theta\colon \mathbb{A}^\sigma \twoheadrightarrow \mathbb{A}/\theta$ (or $\nu_\theta\colon \mathbb{A}^\pi \twoheadrightarrow \mathbb{A}/\theta$, equivalently) is really the canonical extension of $\mu_\theta\colon \mathbb{A} \twoheadrightarrow \mathbb{A}/\theta$, i.e. $(\mu_\theta)^\sigma\colon \mathbb{A}^\sigma \twoheadrightarrow (\mathbb{A}/\theta)^\sigma$, composed with an extra isomorphism to get us from $(\mathbb{A}/\theta)^\sigma$ to $\mathbb{A}/\theta$. But by the results of [8] mentioned above, we know that $(\mu_\theta)^\sigma\colon \mathbb{A}^\sigma \twoheadrightarrow (\mathbb{A}/\theta)^\sigma$ is a DLE-homomorphism, whence $\nu_\theta\colon \mathbb{A}^\sigma \twoheadrightarrow \mathbb{A}/\theta$ must also be a DLE-homomorphism. Now if all the individual $\nu_\theta\colon \mathbb{A}^\sigma \twoheadrightarrow \mathbb{A}/\theta$ are DLE-homomorphisms, it follows that $\nu\colon \mathbb{A}^\sigma \twoheadrightarrow \hat{\mathbb{A}}$ is also a DLE-homomorphism. Thus we have dressed up all the skeletons while keeping the existing DL-homomorphisms $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma, \mathbb{A}^\pi$ and $\nu\colon \mathbb{A}^\sigma\mathbb{A}^\pi \twoheadrightarrow \hat{\mathbb{A}}$ intact. We summarize with Figure 18 on the next page and a theorem.

Theorem 6.4. *Let $\mathbb{A}$ be a bounded distributive lattice expansion. Let $e\colon \mathbb{A} \hookrightarrow \mathbb{A}^\sigma, \mathbb{A}^\pi$ be a canonical embedding. For all $\theta \in \Phi$, define $\mu_\theta\colon \mathbb{A} \twoheadrightarrow \mathbb{A}/\theta$ by*

$$\mu_\theta\colon a \mapsto a/\theta$$

and $\nu_\theta\colon \mathbb{A}^\sigma \twoheadrightarrow \mathbb{A}/\theta$ as the canonical extension of $\mu_\theta\colon \mathbb{A} \twoheadrightarrow \mathbb{A}/\theta$ (modulo isomorphism). Then $\nu\colon \mathbb{A}^\sigma \twoheadrightarrow \hat{\mathbb{A}}$ is a surjection and $\nu e = \mu$, i.e. the homomorphism μ from $\mathbb{A}$ to $\hat{\mathbb{A}}$ can be extended to a surjection ν from $\mathbb{A}^\sigma$ to $\hat{\mathbb{A}}$.

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, hook, "e"] \arrow[dr, "\mu_\theta"'] \arrow[dr, shift left] \arrow[ddr, bend right, "\mu"'] & & \mathbb{A}^\sigma, \mathbb{A}^\pi \arrow[dl, "\nu_\theta"'] \arrow[dl, shift left] \arrow[ddl, bend left, "\nu"] \\
& \boxed{F(\Phi)} & \\
& \hat{\mathbb{A}} \arrow[u, "\hat{\pi}_\theta"] \arrow[u, shift right] &
\end{tikzcd}$$

Figure 18. The lattice expansion homomorphism μ is extended to a surjection ν.

7. Suggestions for further research

What have we achieved? We set out to compare the profinite limit of a lattice expansion to its canonical extension and MacNeille completion. First, we proved that all three constructions do not trivially coincide, and that they already differ in the category of modal algebras. Next, we developed a view on the construction of the profinite limit using duality for modal algebras, where we were able to show that the profinite limit of $\mathbb{A}$, which is constructed from the finite quotients of $\mathbb{A}$, corresponds to a Kripke frame which is constructed from the finite generated subalgebras of the ultrafilter frame of $\mathbb{A}$. Moreover, this special frame is a subframe of the ultrafilter frame of $\mathbb{A}$, thus the profinite limit is a homomorphic image of the canonical extension of $\mathbb{A}$. We then expanded this duality-based view to categories with a similar duality theory, with the category of distributive lattices with operators being the most general example. After that we presented a slightly more complicated technique, combining duality at the lattice level and some additional algebra to prove that even for arbitrary distributive lattice expansions, the profinite limit is a homomorphic image of the canonical extension. Additionally we have shown that the surjection witnessing that the profinite limit is a homomorphic image of the canonical extension is always an extension of the canonical embedding.

This certainly is not all that can be said about our original question concerning the relation between the profinite limit $\hat{\mathbb{A}}$, the MacNeille completion $\bar{\mathbb{A}}$ and the canonical extension $\mathbb{A}^\sigma$ of a lattice expansion. There is at least an indirect connection between $\hat{\mathbb{A}}$ and $\bar{\mathbb{A}}$ since it is shown in [7] that $\mathbb{A}^\sigma$ and $\bar{\mathbb{A}}$ are related.

But where else could we go from here? One option is to try to generalize further. Since our final results apply to arbitrary bounded distributive lattice expansions, there is one road to generalization that we need not explore: arbitrary operations are as general as we can get. The distributive lattice structure does invite us to generalize to arbitrary lattices or partially ordered sets however. We can certainly define the profinite limit of a (non-distributive) lattice, since the construction uses only universal algebra.

Of course generalization is not the only option. Another line of investigation is that of trying to remove all reference to duality from the proofs in Section 6. This reminds us of the fact that we have not provided an algebraic (or maybe category-theoretic) characterization of the profinite limit, like those in Definition 2.12. Other more algebraic questions are the questions when we $\hat{\mathbb{A}} \cong \mathbb{A}$ holds, whether $\widehat{\mu[\mathbb{A}]} \cong \hat{\mathbb{A}}$, and what the relation is between $\widehat{(\mathbb{A}_L)}$ and $(\hat{\mathbb{A}})_L$.

Instead of trying to eliminate duality, we may of course also embrace it, by studying the parallel construction on Kripke frames using finite generated subframes, which might even lead to results directly applicable in modal logic.

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