GENERALIZED POWERLOCALES VIA RELATION LIFTING

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Abstract. This paper introduces an endofunctor V_T on the category of frames, parametrized by an endofunctor T on the category Set that satisfies certain constraints. This generalizes Johnstone's construction of the Vietoris powerlocale, in the sense that his construction is obtained by taking for T the finite covariant power set functor. Our construction of the T-powerlocale $V_T\mathbb{L}$ out of a frame $\mathbb{L}$ is based on ideas from coalgebraic logic and makes explicit the connection between the Vietoris construction and Moss's coalgebraic cover modality.

We show how to extend certain natural transformations between set functors to natural transformations between T-powerlocale functors. Finally, we prove that the operation V_T preserves some properties of frames, such as regularity, zero-dimensionality, and the combination of zero-dimensionality and compactness.

Keywords Locales, frames, Vietoris construction, coalgebra, modal logic, cover modality.

1. Introduction

The aim of this paper[1] is to show how coalgebraic modal logic can be used to understand, study and generalize the point-free topological construction of taking Vietoris powerlocales.

1.1. **Hyperspaces and powerlocales.** The *Vietoris hyperspace construction* is a topological construction on compact Hausdorff spaces, which was introduced by Vietoris (1922) as a generalization of the Hausdorff metric. Given a topological space X one defines a new topology τ_X on $\mathrm{K}\,X$, the set of compact subsets of X. This new topology τ_X has as its basis all sets of the form

$$\nabla\{U_1,\dots,U_n\} := \{F \in \mathrm{K}\,X \mid F \subseteq \textstyle\bigcup_{i=1}^n U_i \text{ and } \forall i \le n,\ F \between U_i\},$$

where $U_1,\dots,U_n \subseteq X$ is a finite collection of open sets and $F \between U$ is notation to indicate that $F \cap U \neq \emptyset$. Alternatively, one can use a subbasis to generate τ_X, consisting of subbasic open sets of the shape

$$\Box U := \{F \in \mathrm{K}\,X \mid F \subseteq U\},$$

and

$$\Diamond U := \{F \in \mathrm{K}\,X \mid F \between U\}.$$

To generate the basic open sets $\nabla\{U_1,\dots,U_n\}$ from $\Box U$ and $\Diamond U$, one can use the following expression:

$$\nabla\{U_1,\dots,U_n\} = \Box\left(\textstyle\bigcup_{i=1}^n U_i\right) \cap \textstyle\bigcap_{i=1}^n \Diamond U_i.$$

In the field of *point-free topology*, a considerable amount of general topology has been recast in a way which makes it more compatible with constructive mathematics and topos theory. (Standard references are Johnstone (1982) and Vickers (1989)). The main idea is to study the lattices of open sets of topological spaces, rather than their associated sets of points. In other words, it is an approach

Date: 21 December 2010; Revised 6 February 2012.

The second author is grateful for financial support from the Netherlands Organization for Scientific Research (NWO) enabling him to make two research visits to Amsterdam. The work of the third author was made possible by NWO VICI grant 639.073.501.

[1]This paper has been accepted for publication in *Mathematical Structures in Computer Science*.

to topology via algebra, where rather than categories of topological spaces, one studies categories of *locales*, or their algebraic counterparts, *frames*. Frames are complete lattices in which finite meets distribute over arbitrary joins, and can be seen as the algebraic models of propositional geometric logic, a branch of logic where finite conjunctions are studied in combination with infinite disjunctions. Substantial parts of this paper arose out of the direct application of techniques from coalgebraic logic to frames/locales. This has led to two consequences: firstly, most results are stated in terms of frames rather than locales, since frames are closer to the Boolean algebras predominantly used in coalgebraic logic. Secondly, we have given little pause to issues of constructivity, in order to be able to directly apply coalgebraic logic techniques. We will briefly revisit these matters in §5. Our bias towards frames notwithstanding, we have favored the name 'powerlocale' over 'powerframe' however.

Johnstone (1982) defines a point-free, syntactic version of the Vietoris powerlocale, using an extension of geometric logic with two unary operators, $\Box$ and $\Diamond$. However he soon also introduces expressions of the shape

$$\Box(\bigvee A) \wedge \bigwedge_{b \in B} \Diamond b,$$

where A and B are finite sets, which should remind the reader of the expression for $\nabla\{U_1, \ldots, U_n\}$ above. Nevertheless, the description of the Vietoris powerlocale is usually given with $\Box$ and $\Diamond$ as primitive, and not without good reason: one may obtain the Vietoris powerlocale by first constructing one-sided locales corresponding to the $\Box$-generators on the one hand and the $\Diamond$-generators on the other, and then joining these two one-sided powerlocales to obtain the Vietoris powerlocale (Vickers & Townsend, 2004). The question remains however, if one can describe the Vietoris powerlocale directly in terms of its basic opens, corresponding to $\nabla\{U_1, \ldots, U_n\}$, rather than the subbasic opens expressed in terms of $\Box$ and $\Diamond$. One of the main contributions of this paper is to show that this is indeed possible.

1.2. **The cover modality and coalgebraic modal logic.** The reader may have noticed that the notation using $\Box$ and $\Diamond$ above is highly suggestive of modal logic. This is no coincidence: Johnstone's presentation of the Vietoris powerlocale in terms of generators and relations extends the axioms of positive (that is, negation-free) modal logic to the geometric setting.

In Boolean-based modal logic, one can define a ∇-modality which is applied to finite *sets* of formulas. This ∇-modality then has the following semantics. If $\mathfrak{M} = \langle W, R, V\rangle$ is a Kripke model and α is a finite set of formulas, then for any state $w \in W$,

$$\begin{aligned} \mathfrak{M}, w \Vdash \nabla\alpha \text{ iff } & \forall a \in \alpha,\ \exists v \in R[w],\ \mathfrak{M}, v \Vdash a \text{ and} \\ & \forall v \in R[w],\ \exists a \in \alpha,\ \mathfrak{M}, v \Vdash a. \end{aligned}$$

In classical modal logic, the ∇-modality is equi-expressive with the $\Box$- and $\Diamond$-modalities, using the following translations:

$$\nabla\alpha \equiv \Box\left(\bigvee\alpha\right) \wedge \bigwedge_{a\in\alpha} \Diamond a,$$

and in the other direction, one can use

$$\Box a \equiv \nabla\{a\} \vee \nabla\emptyset, \text{ and } \Diamond a \equiv \nabla\{a, \top\}.$$

As a primitive modality, ∇ was first introduced by Barwise & Moss (1996) in the study of circularity and by Janin & Walukiewicz (1995) in the study of the modal μ-calculus. It was in Moss's work (Moss, 1999) however that the ∇-modality stepped into the spotlight as a modality suitable for generalization to the abstraction level of *coalgebras*.

The theory of Coalgebra aims to provide a general mathematical framework for the study of state-based evolving systems. Given a endofunctor T on the category Set of sets with functions, a coalgebra of type T, or briefly: a T-coalgebra is simply a function $\sigma\colon X \to TX$, where X is the underlying set of states of the coalgebra, and a T-coalgebra morphism between coalgebras $\sigma\colon X \to TX$ and $\sigma\colon X' \to TX'$ is simply a function $f\colon X \to X'$ such that $Tf \circ \sigma = \sigma' \circ f$. Aczel (1988) introduced T-coalgebras as a means to study transition systems. A natural example of such transition systems is provided by the *Kripke frames* and *Kripke models* used in the model theory of propositional modal logic: the category of Kripke frames and bounded morphisms is isomorphic to the category of P-coalgebras, where $P\colon \mathsf{Set} \to \mathsf{Set}$ is the covariant powerset functor. Universal coalgebra was later introduced by Rutten (2000) as a theoretical framework for modeling behavior of set-based transition systems, parametric in their transition functor $T\colon \mathsf{Set} \to \mathsf{Set}$.

Coalgebraic logics are designed and studied in order to reason formally about coalgebras and their behavior; one of the main applications of this approach is the design of specification and verification languages for coalgebras. The most influential approach to coalgebraic logic, known as *coalgebraic modal logic* (Cîrstea et al., 2009), is to try and generalize propositional modal logic from Kripke structures to the setting of arbitrary set-based coalgebras. Seminal in this approach was the observation of L. Moss in the earlier mentioned paper (Moss, 1999), that the semantics of the cover modality ∇ can be described using the categorical technique of *relation lifting*. This observation paved the way for generalizations to other functors that admit a reasonable notion of relation lifting: Moss introduced a modality ∇_T, parametric in the transition type functor T, which can be interpreted in T-coalgebras via relation lifting.

While Moss's perspective was entirely semantic, his work naturally raised the question whether good derivation systems could be developed for the coalgebraic cover modality ∇_T, parametric in the coalgebra functor T. Building on earlier work by Bílková, Palmigiano & Venema (Palmigiano & Venema, 2007; Bílková et al., 2008) for the power set case, Kupke, Kurz & Venema (2008; 2010) proved soundness and completeness of such a derivation system $\mathbf{M}_T$. The latter paper also introduces, on the category of Boolean algebras, an associated functor $\mathbb{M}_T$, which can be seen as the algebraic correspondent of the topological Vietoris functor on the dual category of Stone spaces.

1.3. **Contribution.** In this paper we translate the coalgebraic modal derivation system $\mathbf{M}_T$ from its Boolean origins (Bílková et al., 2008; Kupke et al., 2008) to the setting of geometric logic. Basically, this means we take some first steps towards developing a *geometric* coalgebraic modal logic, i.e. a logic with finite conjunctions, infinite disjunctions, and the coalgebraic cover modality ∇_T.

The main conceptual contribution of this paper is the introduction of a *generalized powerlocale construction* V_T, parametric in a functor $T\colon \mathsf{Set} \to \mathsf{Set}$ satisfying some categorical conditions. Given a frame $\mathbb{L}$, we define its T-powerlocale $V_T\mathbb{L}$ using a presentation, which takes the set $\{\nabla_T\alpha \mid \alpha \in TL\}$ as generators and the geometric version of the ∇-axioms as relations.

As we will see, the classical Vietoris powerlocale construction is an instantiation of the T-powerlocale, where we take $T = P_\omega$, the covariant finite power set functor. This reveals that the connection between the Vietoris construction and the cover modality, which was implicit in *semantic* form already in (Vietoris, 1922), can also be made explicit *syntactically* using coalgebraic modal logic. Our approach shows how to describe the Vietoris constructions syntactically using the ∇-expressions as primitives, rather than as expressions derived from $\Box$- and $\Diamond$-primitives, as it was introduced in (Johnstone, 1982).

In addition, we prove some technical results concerning the T-powerlocale construction. To start with, we discuss some *functorial properties*; in particular, we show that we are in fact dealing with a *functor*

$$V_T\colon \mathrm{Fr} \to \mathrm{Fr}$$

on the category of frames with algebraic frame homomorphisms. Furthermore, we show how to extend certain natural transformations between transition functors to natural transformations between T-powerlocale functors; this generalizes for instance the frame homomorphism from the Vietoris locale onto the original frame. We also give an alternative *flat site presentation* of the T-powerlocale $V_T\mathbb{L}$, showing that each element of a T-powerlocale has a disjunctive normal form. Finally, we prove some first *preservation results*; in particular, we show that the operation V_T preserves some important properties of frames, such as regularity, zero-dimensionality, and the combination of zero-dimensionality and compactness.

Overview. In §2 we introduce preliminaries on category theory, relation lifting, frame presentations and the classical point-free presentation of the powerlocale. In §3 we introduce the T-powerlocale construction V_T. We then show that the P_ω-powerlocale is isomorphic to the classical Vietoris powerlocale and we discuss some functorial properties of the construction. We conclude this section with providing the above-mentioned flat site presentation of T-powerlocales. In §4 we prove our preservation results, and we provide a new, constructively valid proof of the preservation of compactness for the "classical" Vietoris construction. We finish in §5 with some possibilities for future work.
Acknowledgments. We would like to thank the anonymous referee for providing useful suggestions for improving the presentation of the paper.

2. Preliminaries

2.1. **Basic mathematics.** First we fix some mathematical notation and terminology. Let $f\colon X \to X'$ be a function. Then the *graph* of f is the relation

$$Gr\, f ::= \{(x, f(x)) \in X \times X' \mid s \in X\}.$$

Given a relation $R \subseteq X \times X'$, we denote the *domain* and *range* of R by $\mathsf{dom}(R)$ and $\mathsf{rng}(R)$, respectively. Given subsets $Y \subseteq X$, $Y \subseteq X'$, the *restriction* of R to Y and Y' is given as

$$R{\restriction_{Y \times Y'}} ::= R \cap (Y \times Y').$$

The composition of two relations $R \subseteq X \times X'$ and $R' \subseteq X' \times X''$ is denoted by $R\,;R'$, whereas the composition of two functions $f\colon X \to X'$ and $f'\colon X' \to X''$ is denoted by $f' \circ f$. Thus, we have $Gr\,(f' \circ f) = Gr\, f\,; Gr\, f'$.

We will denote by $P(X)$ and $P_\omega(X)$ the *power set* and *finite power set* of a given set X. The *diagonal on* X is the relation $\Delta_X = \{(x, x) \mid x \in X\}$. Given two sets X, Y we say that X *meets* Y, notation: $X \between Y$, if $X \cap Y$ is inhabited (that is, non-empty).

A *pre-order* is a pair (X, R) where R is a reflexive and transitive relation on X. Given such a pre-order we define the operations $\downarrow_{(X,R)}, \uparrow_{(X,R)} \colon PX \to PX$ by $\downarrow_{(X,R)}(Y) := \{x \in X \mid x\ R\ y$ for some $y \in Y\}$ and $\uparrow_{(X,R)}(Y) := \{x \in X \mid y\ R\ x$ for some $y \in Y\}$. If no confusion is likely, we will write $\downarrow_X$ or $\downarrow$ rather than $\downarrow_{(X,R)}$.

2.2. **Category theory.** We will assume familiarity with the basic notions from category theory, including those of categories, functors, natural transformations, and (co-)monads. As a reference text the reader may consult for instance Mac Lane (1998).

We let Set denote the category with sets as objects and functions as morphism; endofunctors on this category will simply be called *set functors*. The most important set functor that we shall use is the covariant power set functor P, which is in fact (part) of a monad (P, μ, η), with $\eta_X \colon X \to P(X)$ denoting the singleton map $\eta_X \colon x \mapsto \{x\}$, and $\mu_X \colon PPX \to PX$ denoting union, $\mu_X(\mathcal{A}) := \bigcup \mathcal{A}$. The contravariant power set functor will be denoted as $\breve{P}$.

We will restrict our attention to set functors satisfying certain properties, of which the first one is crucial. In order to define it, we need to recall the notion of a (weak) pullback. Given two functions $f_0 \colon X_0 \to X$, $f_1 \colon X_1 \to X$, a *weak pullback* is a set P, together with two functions $p_i \colon P \to X_i$ such that $f_0 \circ p_0 = f_1 \circ p_1$, and in addition, for every triple (Q, q_0, q_1) also satisfying $f_0 \circ q_0 = f_1 \circ q_1$, there is an arrow $h\colon Q \to P$ such that $q_0 = h \circ p_0$ and $q_1 = h \circ p_0$, in a diagram:

$$
\begin{tikzcd}
Q \arrow[dr, dashed, "h"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_0"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_0"] & X_1 \arrow[d, "f_1"] \\
& X_0 \arrow[r, "f_0"'] & X
\end{tikzcd}
$$

For (P, p_0, p_1) to be a *pullback*, we require in addition the arrow h to be unique.

A functor T *preserves weak pullbacks* if it transforms every weak pullback (P, p_0, p_1) for f_0 and f_1 into a weak pullback (TP, Tp_0, Tp_1) for Tf_0 and Tf_1. An equivalent characterization is to require T to *weakly preserve pullbacks*, that is, to turn pullbacks into weak pullbacks. In the next subsection we will see yet another, and motivating, characterization of this property.

The second property that we will impose on our set functors is that of standardness. Given two sets X and X' such that $X \subseteq X'$, let $\iota_{X,X'}$ denote the inclusion map from X into X'. A weak pullback-preserving set functor T is *standard* if it *preserves inclusions*, that is: $T\iota_{X,X'} = \iota_{TX,TX'}$ for every inclusion map $\iota_{X,X'}$.

Remark 2.1. Unfortunately the definition of standardness is not uniform throughout the literature. Our definition of standardness is taken from Moss (1999), while for instance Adámek & Trnková (1990) have an additional condition involving so-called distinguished points. Fortunately, the two definitions are equivalent in case the functor preserves weak pullbacks, see Kupke (2006, Lemma A.2.12).

The restriction to standard functors is not essential, since every set functor is 'almost standard' (Adámek & Trnková, 1990, Theorem III.4.5): given an arbitrary set functor T, we may find a standard set functor T' such that the restriction of T and T' to all non-empty sets and non-empty functions are naturally isomorphic.

Finally, we shall require that our functors are determined by their behavior on finite sets. Call a standard set functor T *finitary* if $TX = \bigcup\{TX' \mid X' \subseteq_\omega X\}$. Our focus on finitary functors is not so much a restriction as a convenient way to express the fact that we are interested in the *finitary version* of an arbitrary set functor, in the sense that P_ω is the finitary version of P. Generally, we may define, for a standard functor T, the functor T_ω that on objects X is defined by $T_\omega X = \bigcup\{TX' \mid X' \subseteq X\}$, while on arrows f we simply put $T_\omega f := Tf$.

Since there are many set functor which are standard, finitary and weak pullback-preserving, the results in this paper have a wide scope.

Example 2.2. The identity functor Id, the finitary power set functor P_ω, and, for each set Q, the constant functor C_Q (given by $C_Q X = Q$ and $C_Q f = id_Q$) are standard, finitary, and preserve weak pullbacks.

For a slightly more involved example, consider the finitary *multiset* functor M_ω. This functor takes a set X to the collection $M_\omega X$ of maps $\mu\colon X \to \mathbb{N}$ of finite support (that is, for which the set $Supp(\mu) := \{x \in X \mid \mu(x) > 0\}$ is finite), while its action on arrows is defined as follows. Given an arrow $f\colon X \to X'$ and a map $\mu \in M_\omega X$, we define $(M_\omega f)(\mu)\colon X' \to \mathbb{N}$ by putting

$$(M_\omega f)(\mu)(x') := \sum\{\mu(x) \mid f(x) = x'\}.$$

With this definition, the functor is not standard, but we may 'standardize' it by representing any map $\mu\colon X \to \mathbb{N}$ of finite support by its 'support graph' $\{(x, \mu x) \mid \mu x > 0\}$. As a variant of M_ω, consider the finitary probability functor D_ω, where $D_\omega X = \{\delta\colon X \to [0,1] \mid Supp(\delta)$ is finite and $\sum_{x\in X} \delta(x) = 1\}$, while the action of D_ω on arrows is just like that of M_ω.

Perhaps more importantly, the class of finitary, standard functors that preserve weak pullbacks, is closed under the following operations: composition ($\circ$) , product ($\times$), co-product ($+$), and exponentiation with respect to some set D ($(\cdot)^D$). As a corollary, inductively define the following class EKPF_ω of *extended finitary Kripke polynomial functors*:

$$T ::= \mathit{Id} \mid P_\omega \mid C_Q \mid M_\omega \mid D_\omega \mid T_0 \circ T_1 \mid T_0 + T_1 \mid T_0 \times T_1 \mid T^D.$$

Then each extended Kripke polynomial functor falls in the scope of the work in this paper.

As running examples in this paper we will often take the *binary tree functor* $B = \mathit{Id} \times \mathit{Id}$, and the finitary power set functor P_ω.

An interesting result of standard functors is that they preserve finite intersections (Adámek & Trnková, 1990, Theorem III.4.6): $T(X \cap Y) = TX \cap TY$. As a consequence, if T is finitary, for any object $\xi \in TX$ we may define

$$\mathit{Base}^T_X(\xi) := \bigcap \{X' \in P_\omega(X) \mid \xi \in TX'\},$$

and show that $\mathit{Base}^T_X(\xi)$ is the *smallest* set X' such that $\xi \in TX'$ (Venema, 2006). In fact, the base maps provide a natural transformation $\mathit{Base}^T : T \to P_\omega$; for referencing we will mention this fact explicitly in the next section.

To facilitate the reasoning in this paper, which will involve objects of various different types, we use a variable naming convention.

Convention 2.3. Let X be a set and let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a functor. We use the following naming convention:

Set	Elements
X	$a, b, \ldots, x, y, \ldots$
TX	$\alpha, \beta, \ldots$
PX	$A, B, \ldots$
PTX	$\Gamma, \Delta, \ldots$
TPX	$\Phi, \Psi, \ldots$

2.3. **Relation lifting.** In §1, we mentioned that coalgebraic modal logic using the cover modality, as introduced by Moss, crucially uses relation lifting, both for its syntax and semantics. Relation lifting is a technique which allows one to extend a functor $T\colon \mathsf{Set} \to \mathsf{Set}$ defined on the category of sets to a functor $\overline{T}\colon \mathsf{Rel} \to \mathsf{Rel}$ on the category of sets and relations in a natural way. In this subsection we will introduce some of the basic facts and definitions about relation lifting.

Let T be a set functor. Given two sets X and X', and a binary relation R between $X \times X'$, we define the *lifted relation* $\overline{T}(R) \subseteq TX \times TX'$ as follows:

$$\overline{T}(R) := \{((T\pi)(\rho), (T\pi')(\rho)) \mid \rho \in TR\},$$

where $\pi\colon R \to X$ and $\pi'\colon R \to X'$ are the projection functions given by $\pi(x, x') = x$ and $\pi'(x, x') = x'$. In a diagram:

$$\begin{tikzcd}
X & R \arrow[l, "\pi"'] \arrow[r, "\pi'"] & X'
\end{tikzcd}$$

$$\begin{tikzcd}
TX & TR \arrow[l, "T\pi"'] \arrow[r, "T\pi'"] \arrow[d] \arrow[dd, bend right, "\langle T\pi, T\pi' \rangle"'] & TX' \\
 & \overline{T}R \arrow[d, hook] & \\
 & TX \times TX' \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

In other words, we apply the functor T to the relation R, seen as a *span*

$$X \xleftarrow{\pi} R \xrightarrow{\pi'} X' ,$$

and define $\overline{T}R$ is the image of TR under the product map $\langle T\pi, T\pi'\rangle$ obtained from the lifted projection maps $T\pi$ and $T\pi'$.

Let us first see some concrete examples.

Example 2.4. Fix a relation $R \subseteq X \times X'$. For the identity and constant functors, we find, respectively:

$$\begin{array}{rcl} \overline{Id}R & = & R \\ \overline{C_Q}R & = & \Delta_Q. \end{array}$$

The relation lifting associated with the power set functor P can be defined concretely as follows:

$$\overline{P}R = \{(A, A') \in PX \times PX' \mid \forall a \in A\, \exists a' \in A'.aRa' \text{ and } \forall a' \in A'\, \exists a \in A.aRa'\}.$$

This relation is known under many names, of which we mention that of the *Egli-Milner* lifting of R. For any standard, weak pullback preserving functor T it can be shown (Kupke et al., 2010) that the lifting of T_ω agrees with that of T, in the sense that $\overline{T_\omega}R = \overline{T}R \cap (T_\omega X \times T_\omega X')$. From this it follows that

$$\text{for all } A \in T_\omega X, A' \in T_\omega X' : A\ \overline{P_\omega}R\ A' \text{ iff } A\ \overline{P}R\ A',$$

and for this reason, we shall write $\overline{P}R$ rather than $\overline{P_\omega}R$.

Relation lifting for the finitary multiset functor is slightly more involved: given two maps $\mu \in M_\omega X, \mu' \in M_\omega X'$, we put

$$\begin{array}{l} \mu\ \overline{M_\omega}R\ \mu' \text{ iff there is some map } \rho\colon R \to \mathbb{N} \text{ such that} \\ \qquad\qquad \forall x \in X. \sum\{\rho(x, x') \mid x' \in X'\} = 1, \text{ and} \\ \qquad\qquad \forall x' \in X'. \sum\{\rho(x, x') \mid x \in X\} = 1. \end{array}$$

The definition of $\overline{D_\omega}$ is similar.

Finally, relation lifting interacts well with various operations on functors (Hermida & Jacobs, 1998). In particular, we have

$$\begin{array}{rcl} \overline{T_0 \circ T_1}R & = & \overline{T}_0(\overline{T}_1 R) \\ \overline{T_0 + T_1}R & = & \overline{T}_0 R \cup \overline{T}_1 R \\ \overline{T_0 \times T_1}R & = & \left\{((\xi_0, \xi_1), (\xi'_0, \xi'_1)) \mid (\xi_i, \xi'_i) \in \overline{T}_i, \text{ for } i \in \{0, 1\}\right\}. \\ \overline{T^D}R & = & \{(\varphi, \varphi') \mid (\varphi(d), \varphi'(d) \in \overline{T}R \text{ for all } d \in D\} \end{array}$$

Remark 2.5. Strictly speaking, the definition of the relation lifting of a given relation R depends on the type of the relation, i.e. given sets X, X', Y, Y' such that $R \subseteq X \times X'$ and $R \subseteq Y \times Y'$, it matters whether we look at R as a relation from X to X' or as a relation from Y to Y'. We have avoided this potential source of ambiguity by requiring the functor T to be *standard*, see Fact 2.6(6).

Relation lifting has a number of properties that we will use throughout the paper. It can be shown that relation lifting interacts well with the operation of taking the graph of a function $f : X \to X'$, and with most operations on binary relations. Most of the properties below are easy to establish — we refer to (Kupke et al., 2010) for proofs.

Fact 2.6. *Let T be a set functor. Then the relation lifting $\overline{T}$ satisfies the following properties, for all functions $f\colon X \to X'$, all relations $R, S \subseteq X \times X'$, and all subsets $Y \subseteq X$, $Y' \subseteq X'$:*

(1) *$\overline{T}$ extends T: $\overline{T}(Gr\, f) = Gr\,(Tf)$;*
(2) *$\overline{T}$ preserves the diagonal: $\overline{T}(\Delta_X) = \Delta_{TX}$;*
(3) *$\overline{T}$ commutes with relation converse: $\overline{T}(R^{\smile}) = (\overline{T}R)^{\smile}$;*

(4) $\overline{T}$ *is monotone: if* $R \subseteq S$ *then* $\overline{T}(R) \subseteq \overline{T}(S)$*;*
(5) $\overline{T}$ *distributes over composition:* $\overline{T}(R\,;S) = \overline{T}(R)\,;\overline{T}(S)$*, if* T *preserves weak pullbacks.*
(6) $\overline{T}$ *commutes with restriction:* $\overline{T}(R{\restriction}_{Y \times Y'}) = \overline{T}R{\restriction}_{TY \times TY'}$*, if* T *is standard and preserves weak pullbacks.*

Fact 2.6(5) plays a key role in our work. In fact, distributivity of $\overline{T}$ over relation composition is *equivalent* to T preserving weak-pullbacks; the proof of this equivalence goes back to Trnková (1977).

Many proofs in this paper will be based on Fact 2.6, and we will not always provide all technical details. In the lemma below we have isolated some facts that will be used a number of times; the proof may serve as a sample of an argument using properties of relation lifting.

Lemma 2.7. *Let* $T\colon \mathsf{Set} \to \mathsf{Set}$ *be a standard, finitary, weak pullback-preserving functor. Let* X, Y *be sets, let* $f, g\colon X \to Y$ *be two functions and let* $R \subseteq X \times X$ *and* $S \subseteq Y \times Y$ *be relations.*
(1) *If* (X, R) *is a pre-order, then so is* $(TX, \overline{T}R)$*.*
(2) *If* $f(x)\ S\ g(x)$ *for all* $x \in X$*, then* $Tf(\alpha)\ \overline{T}S\ Tg(\alpha)$ *for all* $\alpha \in TX$*.*
(3) *If* $x\ R\ y$ *implies* $f(x)\ S\ g(y)$ *for all* $x, y \in X$*, then* $\alpha\ \overline{T}R\ \beta$ *implies* $(Tf)\alpha\ \overline{T}S\ (Tg)\beta$ *for all* $\alpha, \beta \in TX$*.*

Proof. For part 1, observe that (X, R) is a pre-order iff $\Delta_X \subseteq R$ and $R\,;R \subseteq R$. Hence, if (X, R) is a pre-order, it follows from Fact 2.6(2,4) that $\Delta_{TX} = \overline{T}\Delta_X \subseteq \overline{T}R$, and from Fact 2.6(5,4) that $\overline{T}R\,;\overline{T}R = \overline{T}(R\,;R) \subseteq \overline{T}R$, implying that $(TX, \overline{T}R)$ is a pre-order as well.

For part 2, observe that the antecedent can be succinctly expressed as

$$(\mathit{Gr}\,f)^{\smile}\,;\mathit{Gr}\,g \subseteq S.$$

Then it follows by the properties of relation lifting that

$$\begin{aligned}
(\mathit{Gr}\,Tf)^{\smile}\,;\mathit{Gr}\,Tg &= (\overline{T}(\mathit{Gr}\,f))^{\smile}\,;\overline{T}(\mathit{Gr}\,g) && \text{(Fact 2.6(1))}\\
&= \overline{T}((\mathit{Gr}\,f)^{\smile})\,;\overline{T}(\mathit{Gr}\,g) && \text{(Fact 2.6(3))}\\
&= \overline{T}((\mathit{Gr}\,f)^{\smile}\,;\mathit{Gr}\,g) && \text{(Fact 2.6(5))}\\
&\subseteq \overline{T}S && \text{(Fact 2.6(4))}
\end{aligned}$$

But the inclusion $(\mathit{Gr}\,Tf)^{\smile}\,;\mathit{Gr}\,Tg \subseteq \overline{T}S$ is just another way of stating the conclusion of part 2.

For part 3, we reformulate the statement of its antecedent as

$$(\mathit{Gr}\,f)^{\smile}\,;R\,;\mathit{Gr}\,g \subseteq S.$$

On the basis of this we may reason, via a completely analogous argument to the one just given, that

$$(\mathit{Gr}\,Tf)^{\smile}\,;\overline{T}R\,;\mathit{Gr}\,Tg \subseteq TS,$$

which is equivalent way of phrasing the conclusion of part 3. □

Relation lifting interacts with the map Base^T as follows (see Kupke et al., 2010):

Fact 2.8. *Let* T *be a standard, finitary, weak pullback-preserving functor.*
(1) Base^T *is a natural transformation* $\mathit{Base}^T\colon T \to P_\omega$*. That is, given a map* $f\colon X \to X'$ *the following diagram commutes:*

$$\begin{tikzcd}
TX \arrow[r, "\mathit{Base}^T_X"] \arrow[d, "Tf"'] & P_\omega X \arrow[d, "Pf"] \\
TX' \arrow[r, "\mathit{Base}^T_{X'}"] & P_\omega X'
\end{tikzcd}$$

(2) *Given a relation* $R \subseteq X \times X'$ *and elements* $\alpha \in TX$*,* $\beta \in TY$*, it follows from* $\alpha\ \overline{T}R\ \beta$ *that* $\mathit{Base}^T(\alpha)\ \overline{P}R\ \mathit{Base}^T(\beta)$*.*

An interesting relation to which we shall apply relation lifting is the *membership* relation $\in$. If needed, we will denote the membership relation restricted to a given set X as the relation $\in_X \subseteq X \times PX$. Given a set X and $\Phi \in TPX$, we define

$$\lambda^T_X(\Phi) = \{\alpha \in TX \mid \alpha \mathrel{\overline{T}\!\!\in_X} \Phi\}.$$

Elements of $\lambda^T(\Phi)$ will be called *lifted members of* Φ. Properties of λ^T are intimately related to those of $\overline{T}$ (see Kupke et al., 2010):

Fact 2.9. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor. Then the collection of maps λ^T_X forms a distributive law with respect to both the co- and the contravariant power set functor. That is, λ^T provides two natural transformations, $\lambda^T : TP \to PT$, and $\lambda^T : T\breve{P} \to \breve{P}T$.*

Remark 2.10. One can strengthen Fact 2.9: λ^T is actually a distributive law over the *monad* (P, μ, η), in the sense of being also compatible with the unit η and the multiplication μ of P, as given by the following two diagrams:

$$\begin{tikzcd}
TX \arrow[r, "T\eta_X"] \arrow[dr, "\eta_{TX}"'] & TPX \arrow[d, "\lambda^T_X"] \\
& PTX
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TPPX \arrow[r, "\lambda^T_{PX}"] \arrow[d, "T\mu_X"] & PTPX \arrow[r, "P\lambda^T_X"] & PPTX \arrow[d, "\mu_X"] \\
TPX \arrow[rr, "\lambda^T_X"'] & & PTX
\end{tikzcd}$$

In the terminology of Street (1972), (T, λ^T) is a *monad opfunctor* from the monad P to itself, and there is a one-one correspondence between the monad opfunctors and the functors T equipped with extensions to endofunctors on the *Kleisli category* $\mathsf{Kl}(P)$ associated with P. (The explicit results in (Street, 1972), using the 2-functor $\mathbf{Alg_C}$, are in terms of monad functors and extensions to the category of Eilenberg-Moore algebras. The results for monad opfunctors and the Kleisli category are dual.) Note that the Kleisli category of the power set monad is (isomorphic to) the category Rel with sets as objects, and binary relations as arrows. The correspondence mentioned then links the natural transformation λ^T to the notion of relation lifting $\overline{T}$.

Lemma 2.11. *Let T be a standard, finitary, weak pullback-preserving functor. Let X be some set and let $\Phi \in TPX$.*

1. *If $\varnothing \in Base^T(\Phi)$ then $\lambda^T(\Phi) = \varnothing$.*
2. *If $Base^T(\Phi)$ consists of singletons only, then $\lambda^T(\Phi)$ is a singleton.*
3. *If T maps finite sets to finite sets, then for all $\Phi \in TP_\omega X$, $|\lambda^T(\Phi)| < \omega$.*

Proof. For part 1, suppose that α is a lifted member of Φ; then we may derive by Fact 2.8 that $Base^T(\alpha) \mathrel{\overline{P}\!\!\in} Base^T(\Phi)$. But from this it would follow, if $\varnothing \in Base^T(\Phi)$, that $Base^T(\alpha)$ contains a member of $\varnothing$, which is clearly impossible. Consequently, then $\lambda^T(\Phi)$ is empty.

For part 2, observe that another way of saying that $Base^T(\Phi)$ consists of singletons only, is that $\Phi \in TS_X$, with $S_X := \{\{x\} \mid x \in X\}$. Let $\theta_X : S_X \to X$ be the inverse of η_X, that is, θ_X is the bijection mapping a singleton $\{x\}$ to its unique member x. Clearly then, we have $(Gr\,\theta_X)\breve{} = \in\restriction_{X \times S_X}$, from which it follows by Fact 2.6 that $(Gr\,T\theta_X)\breve{} = \overline{T}\!\!\in\restriction_{TX \times TS_X}$. From this it is immediate that if $\Phi \in TS_X$, then $(T\theta_X)(\Phi)$ is the unique lifted member of Φ.

Finally, we consider part 3. Since T is finitary, $\Phi \in TP_\omega X$ implies that $\Phi \in TP_\omega Y$ for some finite set Y, and from this it follows that $Base^T(\Phi) \subseteq P_\omega Y$. If α is a lifted member of Φ, then by Fact 2.8 we obtain $Base^T(\alpha) \mathrel{\overline{P}\!\!\in} Base^T(\Phi)$, and so in particular we find $Base^T(\alpha) \subseteq \bigcup Base^T(\Phi) \subseteq Y$. From this it follows that $\lambda^T(\Phi) \subseteq TY$, and so $\lambda^T(\Phi)$ must be finite by the assumption on T. $\square$

2.4. **Frames and their presentations.** A *frame* is a complete lattice in which finite meets distribute over arbitrary joins. The signature of frames consists of arbitrary joins and finite meets, and it will be convenient for us to include the top and bottom as well. Thus a frame will usually be given as $\mathbb{L} = \langle L, \bigvee, \wedge, 0, 1\rangle$, while we will often consider join and meet as functions $\bigvee_{\mathbb{L}} : PL \to L$ and

$\bigwedge_{\mathbb{L}} \colon P_\omega L \to L$. This enables us for instance to define a frame homomorphism $f \colon \mathbb{L} \to \mathbb{M}$ as a map from L to M satisfying $f \circ \bigwedge = \bigwedge \circ (P_\omega f)$ and $f \circ \bigvee = \bigvee \circ (Pf)$. By Fr we denote the category of frames and frame homomorphisms. The initial frame (the lattice of truth values) will be denoted as Ω, and for a given frame $\mathbb{L}$ we will let $!_{\mathbb{L}}$ denote the unique frame homomorphism from Ω to $\mathbb{L}$, omitting the subscript if $\mathbb{L}$ is clear from context.

The order relation $\leq_{\mathbb{L}}$ of a frame $\mathbb{L}$ is given by $a \leq_{\mathbb{L}} b$ if $a \wedge b = a$ (or, equivalently, $a \vee b = b$). We can adjoin an implication operation to a frame $\mathbb{L}$ by defining $a \to b := \bigvee\{c \mid a \wedge c \leq b\}$; this operation turns $\mathbb{L}$ into a Heyting algebra. As a special case of implication we can consider the *negation*: $\neg a := \bigvee\{c \mid a \wedge c = 0\}$. Generally, neither of these two operations is preserved by frame homomorphisms. A subset S of $\mathbb{L}$ is *directed* if for every $s_0, s_1 \in S$ there is an element $s \in S$ such that $s_0, s_1 \leq s$. The join of a directed set S is often denoted as $\bigvee^{\uparrow} S$.

A *frame presentation* is a tuple $\langle G \mid R\rangle$ where G is a set of generators and $R \subseteq PP_\omega G \times PP_\omega G$ is a set of relations. A presentation $\langle G \mid R\rangle$ *presents* a frame $\mathbb{L}$ if there exists a function $f \colon G \to L$ which is *compatible with* R, i.e. such that

$$\text{for all } (t_1, t_2) \in R,\ \bigvee_{A \in t_1} \bigwedge (P_\omega f)A = \bigvee_{B \in t_2} \bigwedge (P_\omega f)B,$$

and for all frames $\mathbb{M}$ and functions $g \colon G \to M$ compatible with R, there is a unique frame homomorphism $g' \colon \mathbb{L} \to \mathbb{M}$ such that $g'f = g$. We call f the *insertion of generators* (of G in $\mathbb{L}$).

Fact 2.12. *Every frame presentation presents a frame.*

The details of the proof of the above fact (found in Vickers, 1989, §4.4) tell us how to construct a unique frame given a presentation $\langle G \mid R\rangle$. Omitting these details of the construction, we denote this unique frame by $\mathrm{Fr}\langle G \mid R\rangle$. We will usually write $\bigvee_{i \in I} \bigwedge A_i = \bigvee_{j \in J} \bigwedge B_j$ instead of $(\{A_i \mid i \in I\}, \{B_j \mid j \in J\})$ when specifying relations. In light of the fact that $a \leq b$ iff $a \vee b = b$, we will also allow ourselves the liberty to specify inequalities of the shape $\bigvee_{i \in I} \bigwedge A_i \leq \bigvee_{j \in J} \bigwedge B_j$ as relations. It follows from the proof of Fact 2.12 that if $f \colon G \to \mathrm{Fr}\langle G \mid R\rangle$ is the insertion of generators, then every element of $\mathrm{Fr}\langle G \mid R\rangle$ can be written as $\bigvee_{i \in I} \bigwedge P_\omega f A$ for some $\{A_i \mid i \in I\} \in PP_\omega G$; in other words every element of $\mathrm{Fr}\langle G \mid R\rangle$ can be written as an infinite disjunction of finite conjunctions of generators.

We will now introduce flat site presentations for frames, which have as one of their main advantages that they allow us to assume that an arbitrary element of the frame being presented is an infinite join of generators. A *flat site* is a triple $\langle X, \sqsubseteq, \lhd_0\rangle$, where $\langle X, \sqsubseteq\rangle$ is a pre-order and $\lhd_0 \subseteq X \times PX$ is a binary relation such that for all $b \sqsubseteq a \lhd_0 A$, there exists $B \subseteq {\downarrow}A \cap {\downarrow}b$ such that $b \lhd_0 B$. A flat site $\langle X, \sqsubseteq, \lhd_0\rangle$ presents a frame $\mathbb{L}$ if there exists a function $f \colon X \to L$ such that

- f is order-preserving,
- $1 \leq \bigvee (Pf)X$,
- for all $a, b \in X$, $f(a) \wedge f(b) \leq \bigvee (Pf)({\downarrow}a \cap {\downarrow}b)$, and
- for all $a \lhd_0 A$, $f(a) \leq \bigvee (Pf)A$

and for all frames $\mathbb{M}$ and all $g \colon X \to M$ satisfying the above two properties, there exists a unique frame homomorphism $g' \colon \mathbb{L} \to \mathbb{M}$ such that $g' \circ f = g$. Specifically, for all $a \in \mathbb{L}$,

$$g'(a) = \bigvee\{g(x) \mid f(x) \leq a\}.$$

To put it another way, the frame presented by a flat site is

$$\begin{aligned}\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0\rangle \simeq \mathrm{Fr}\langle X \mid \quad & a \leq b \quad (a \sqsubseteq b),\\ & a \leq \bigvee A \quad (a \lhd_0 A),\\ & 1 = \bigvee X\\ & a \wedge b = \bigvee\{c \mid c \sqsubseteq a, c \sqsubseteq b\}\rangle.\end{aligned}$$

A *suplattice* is a complete $\bigvee$-semilattice; accordingly, a suplattice homomorphism is a map which preserves $\bigvee$. A *suplattice presentation* is a triple $\langle X, \sqsubseteq, \lhd_0\rangle$ where $\langle X, \sqsubseteq\rangle$ is a pre-order and $\lhd_0 \subseteq$

$X \times PX$. A suplattice presentation $\langle X, \sqsubseteq, \lhd_0 \rangle$ presents a suplattice $\mathbb{L}$ if there exists a function $f \colon X \to L$ such that

- f is order-preserving;
- for all $a \lhd_0 A$, $f(a) \leq \bigvee Pf(A)$;

and for all suplattices $\mathbb{M}$ and all functions $g \colon X \to M$ respecting the above two conditions, there exists a unique suplattice homomorphism $g' \colon \mathbb{L} \to \mathbb{M}$ such that $g' \circ f = g$. Every suplattice presentation presents a suplattice (Jung et al., 2008, Prop. 2.5). Now observe that every flat site can also be seen as a suplattice presentation with an additional stability condition. Consequently, given a flat site $\langle X, \sqsubseteq, \lhd_0 \rangle$, we can generate two different objects with it: a frame $\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0 \rangle$ and a suplattice $\mathrm{SupLat}\langle X, \sqsubseteq, \lhd_0 \rangle$. The Flat site Coverage Theorem (Vickers, 2006, Theorem 5) tells us that these two objects are in fact order isomorphic.

Fact 2.13. *Let $\langle X, \sqsubseteq, \lhd_0 \rangle$ be a flat site. Then* $\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0 \rangle \simeq \mathrm{SupLat}\langle X, \sqsubseteq, \lhd_0 \rangle$.

We record the following consequences of the above fact. Suppose that $\langle X, \sqsubseteq, \lhd_0 \rangle$ is a flat site which presents a frame $\mathbb{L}$ via $f \colon X \to L$. Then

- every element of $\mathbb{L}$ is of the shape $\bigvee Pf(A)$ for some $A \in PX$;
- we can use $\langle X, \sqsubseteq, \lhd_0 \rangle$ both to define suplattice homomorphisms and frame homomorphisms.

2.5. **Powerlocales via $\Box$ and $\Diamond$.** We will now introduce the Vietoris powerlocale. In line with our generally algebraic approach we shall define it directly as a functor on the category of frames rather than its opposite, the category of locales. In its full generality it originates (as the "Vietoris construction") in (Johnstone, 1985), with some earlier, more restricted references in (Johnstone, 1982). For locales it is a localic analogue of hyperspace (with Vietoris topology). The points are (in bijection with) certain sublocales of the original locale. For a full constructive description see (Vickers, 1997).

Given a frame $\mathbb{L}$, we first define $L_\Box := L$ and $L_\Diamond := L$, and then

$$
\begin{array}{rll}
V\mathbb{L} := \mathrm{Fr}\langle L_\Box \oplus L_\Diamond \mid & \Box 1 = 1 & \\
& \Box(a \wedge b) = \Box a \wedge \Box b & \\
& \Box(\bigvee^{\uparrow} A) = \bigvee^{\uparrow}_{a \in A} \Box a & (A \in PL \text{ directed}) \\
& \Diamond(\bigvee A) = \bigvee_{a \in A} \Diamond a & (A \in PL) \\
& \Box a \wedge \Diamond b \leq \Diamond(a \wedge b) & \\
& \Box(a \vee b) \leq \Box a \vee \Diamond b & \\
& \rangle &
\end{array}
$$

Remark 2.14. We are abusing notation when specifying the relations in the definition above. Strictly speaking, we have two maps, $\Box \colon L_\Box \to V\mathbb{L}$ for the left copy of $\mathbb{L}$ and $\Diamond \colon L_\Diamond \to V\mathbb{L}$ for the right copy of $\mathbb{L}$, so that the insertion of generators is the map $\Box \oplus \Diamond \colon L_\Box \oplus L_\Diamond \to V\mathbb{L}$.

Johnstone (1985) shows that V gives a monad on the category of locales, i.e. a comonad on the category of frames. We shall not need the full strength of this here, but some of the ingredients of the comonad structure are easy to check.

- V is functorial. If $f \colon \mathbb{L} \to \mathbb{M}$ is a frame homomorphism, then the function $(\Box f) \oplus (\Diamond f) \colon L_\Box \oplus L_\Diamond \to VM$ is compatible with the relations in the presentation of $V\mathbb{L}$, so that there is a frame homomorphism $Vf \colon V\mathbb{L} \to V\mathbb{M}$ extending this map. It is also easy to show functoriality.
- The counit $i_{\mathbb{L}} \colon V\mathbb{L} \to \mathbb{L}$ is given by $\Box a \mapsto a$ and $\Diamond a \mapsto a$. The comultiplication $\mu_{\mathbb{L}} \colon V\mathbb{L} \to VV\mathbb{L}$ is given by $\Box a \mapsto \Box\Box a$ and $\Diamond a \mapsto \Diamond\Diamond a$.

3. The T-powerlocale construction

In this section we arrive at the main conceptual contribution of this paper. Given a weak pullback-preserving, standard, finitary functor $T \colon \mathsf{Set} \to \mathsf{Set}$, we define its associated T-powerlocale functor $V_T \colon \mathsf{Fr} \to \mathsf{Fr}$ on the category of frames, using the Carioca axioms for coalgebraic modal logic. This construction truly generalizes the Vietoris powerlocale construction, because we will see that the P_ω-powerlocale is isomorphic to the Vietoris powerlocale. The other two major results in this section are the fact that one can lift a natural transformation between transition functors $\rho\colon T' \to T$ to a natural transformation $\widehat{\rho}\colon V_T \to V_{T'}$ going in the other direction, and the fact that T-powerlocales are join-generated by their generators of the shape $\nabla\alpha$. We will establish the latter fact via the stronger result by showing that $V_T\mathbb{L}$ admits a flat site presentation. The fact that $V_T\mathbb{L}$ is join-generated by its generators is not entirely surprising, since the Carioca axioms were designed with the desirability of conjunction-free disjunctive normal forms in mind (Bílková et al., 2008); however the precise mathematical formulation of this property, using flat sites and suplattices, is an improvement over what was previously known.

This section is organized as follows. In §3.1 we introduce the T-powerlocale construction on frames. In §3.2 we make technical observations about T-powerlocales. In §3.3, we consider two instantiations of the T-powerlocale construction, the most notable of which is the P_ω-powerlocale which is isomorphic to the classical Vietoris powerlocale. In §3.4 we extend the T-powerlocale construction to a functor V_T on the category of frames, and we show how one can lift natural transformations between set functors T, T' to natural transformations between powerlocale functors V_T, $V_{T'}$. We conclude this section with §3.5, in which we show that the T-powerlocale construction admits a flat site presentation, a corollary of which is that each element of $V_T\mathbb{L}$ has a disjunctive normal form.

3.1. **Introducing the T-powerlocale.** In this subsection, we will use the Carioca axioms for coalgebraic modal logic (Bílková et al., 2008) to define the T-powerlocale $V_T\mathbb{L}$ of a given frame $\mathbb{L}$ using a frame presentation, i.e. using generators and relations. The generators of $V_T\mathbb{L}$ will be given by the set TL; in order to specify the relations we will use *relation lifting* (§2.3) and *slim redistributions*, which we will introduce below. In addition, we will provide an alternative presentation of $V_T\mathbb{L}$, which does not use slim redistributions. From a conceptual viewpoint, it is not immediately obvious which presentation of $V_T\mathbb{L}$ should be taken as the primary definition. Our choice to use slim redistributions in the primary definition is motivated by the extant literature (Bílková et al., 2008; Kupke et al., 2008, 2010).

Definition 3.1. Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor, let X be a set and let $\Gamma \in P_\omega TX$. The set of all *slim redistributions* of Γ is defined as follows:

$$SRD(\Gamma) = \left\{\Psi \in TP_\omega\left(\bigcup\nolimits_{\gamma\in\Gamma} Base^T(\gamma)\right) \mid \forall\gamma\in\Gamma,\, \gamma\,\overline{T}{\in}\,\Psi\right\}$$

Intuitively, $\Psi \in TP_\omega X$ is a slim redistribution of $\Gamma \in P_\omega TX$ if (i) Ψ is 'obtained from the material of Γ', that is:

$$\Psi \in TP_\omega\left(\bigcup\nolimits_{\gamma\in\Gamma} Base^T(\gamma)\right),$$

and (ii) every element of Γ is a lifted member of Ψ, or equivalently, $\Gamma \subseteq \lambda^T(\Psi)$. We illustrate this with the motivating example of slim redistributions, namely slim redistribution for the finite powerset functor.

Example 3.2. Recall from Example 2.4 that if $R \subseteq X \times Y$ is a relation then $\overline{P_\omega}R \subseteq P_\omega X \times P_\omega Y$ can be characterized as follows:

$$\alpha\overline{P_\omega}R\beta \text{ iff } \forall x\in\alpha,\,\exists y\in\beta,\, xRy \text{ and } \forall y\in\beta,\,\exists x\in\alpha,\, xRy.$$

In particular, for $\in\; \subseteq X \times PX$ we get $\alpha\;\overline{P_\omega}{\in}\;\Gamma$ iff $\alpha \subseteq \bigcup\Gamma$ and $\forall\gamma\in\Gamma,\,\gamma \between \alpha$. (Recall that $\gamma \between \alpha$ means that $\gamma\cap\alpha$ is inhabited.) For an order $\leq$, let us define the *upper, lower* and *convex* pre-orders on finite sets:

$$\begin{aligned}
\alpha \leq_L \beta &\text{ if } \alpha \subseteq {\downarrow}\beta, \text{ i.e. } \forall x\in\alpha,\,\exists y\in\beta,\, x\leq y\\
\alpha \leq_U \beta &\text{ if } {\uparrow}\alpha \supseteq \beta, \text{ i.e. } \forall y\in\beta,\,\exists x\in\alpha,\, x\leq y\\
\alpha \leq_C \beta &\text{ if } \alpha \leq_L \beta \text{ and } \alpha \leq_U \beta.
\end{aligned}$$

Thus $\overline{P_\omega}\leq$ is $\leq_C$.

Next, if $\alpha \in P_\omega S$ then

$$Base(\alpha) = \bigcap\{S' \in P_\omega(S) \mid \alpha \subseteq S'\} = \alpha.$$

From this, if $\Gamma \in P_\omega P_\omega X$ then

$$\begin{aligned} SRD(\Gamma) &= \{\Psi \in P_\omega P_\omega\left(\bigcup\Gamma\right) \mid \forall\gamma \in \Gamma,\, (\gamma \subseteq \bigcup\Psi \text{ and } \forall\alpha \in \Psi,\, \alpha \between \gamma) \\ &= \{\Psi \in P_\omega P_\omega\left(X\right) \mid \bigcup\Psi = \bigcup\Gamma \text{ and } \forall\gamma \in \Gamma,\, \forall\alpha \in \Psi,\, \alpha \between \gamma\}. \end{aligned}$$

Definition 3.3. Let T be a standard, finitary, weak pullback-preserving functor. Let $\mathbb{L}$ be a frame. We define the *T-powerlocale of* $\mathbb{L}$

$$V_T\mathbb{L} := \mathrm{Fr}\langle TL \mid (\nabla 1), (\nabla 2), (\nabla 3)\rangle,$$

where the relations are the *Carioca axioms* (Bílková et al., 2008):

$$\begin{array}{lll} (\nabla 1) & \nabla\alpha \leq \nabla\beta, & (\alpha \mathrel{\overline{T}{\leq}} \beta) \\ (\nabla 2) & \bigwedge_{\alpha\in\Gamma}\nabla\alpha \leq \bigvee\{\nabla(T\bigwedge)\Psi \mid \Psi \in SRD(\Gamma)\}, & (\Gamma \in P_\omega TL) \\ (\nabla 3) & \nabla(T\bigvee)\Phi \leq \bigvee\{\nabla\beta \mid \beta \mathrel{\overline{T}{\in}} \Phi\}, & (\Phi \in TPL) \end{array}$$

Remark 3.4. To be precise, we assume that $\nabla\colon TL \to V_TL$ is the insertion of generators, so when specifying the relations we should write e.g. $\alpha \leq \beta$ instead of $\nabla\alpha \leq \nabla\beta$. The way we have specified the relations above is more consistent with (Bílková et al., 2008).

We will discuss the instantiation of these axioms for $T = P_\omega$ in some more detail in §3.3.

We will now present a very useful equivalent definition of $V_T\mathbb{L}$. The crucial observation behind the alternative definition of $V_T\mathbb{L}$ is the following technical lemma, which characterizes the slim redistributions of a given finite subset Γ of $\langle TL, \overline{T}{\leq}\rangle$ as the maximal lower bounds of Γ. Observe that the lemma also holds in case $\Gamma = \emptyset$.

Lemma 3.5. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor, let $\mathbb{L}$ be a meet-semilattice (e.g., a frame) and let $\Gamma \in P_\omega TL$. Then for any $\alpha \in TL$, the following are equivalent:*

(a) *$\alpha \in TL$ is a lower bound of Γ, that is, $\alpha \mathrel{\overline{T}{\leq}} \gamma$ for all $\gamma \in \Gamma$;*

(b) *$\alpha \mathrel{\overline{T}{\leq}} (T\bigwedge)\Phi$ for some $\Phi \in SRD(\Gamma)$.*

In particular, if $\Phi \in SRD(\Gamma)$ then $(T\bigwedge)\Phi \mathrel{\overline{T}{\leq}} \gamma$ for all $\gamma \in \Gamma$.

Proof. Recall that

$$SRD(\Gamma) := \left\{\Psi \in TP\left(\bigcup_{\gamma\in\Gamma} Base^T(\gamma)\right) \mid \Gamma \subseteq \lambda^T(\Psi)\right\}.$$

For the implication from (b) to (a), observe that for any $a \in L$ and $A \in P_\omega L$, we have that $a \in A$ implies that $\bigwedge A \leq a$. By Fact 2.6 it follows that for all $\gamma \in TL$ and $\Psi \in TP_\omega L$, if $\gamma \mathrel{\overline{T}{\in}} \Psi$ then $T\bigwedge(\Psi) \mathrel{\overline{T}{\leq}} \gamma$. Now suppose that Ψ is a slim redistribution of Γ. Then $\Gamma \subseteq \lambda^T(\Psi)$, and so $(T\bigwedge)\Psi$ is a $\overline{T}{\leq}$-lower bound of Γ. From this the implication (b) $\Rightarrow$ (a) is immediate.

For the opposite implication, take $\alpha \in TL$ such that $\forall\gamma \in \Gamma$, $\alpha \mathrel{\overline{T}{\leq}} \gamma$. Then by Fact 2.8, we obtain $Base^T(\alpha) \mathrel{\overline{P}{\leq}} Base^T(\gamma)$ for all $\gamma \in \Gamma$. Abbreviate $C := \bigcup_{\gamma\in\Gamma} Base^T(\gamma)$, and define $f\colon Base^T(\alpha) \to PC$ as follows:

$$f\colon a \mapsto {\uparrow_L} a \cap C,$$

that is: $f(a) = \{c \in C \mid a \leq c\}$. Then Tf is a function

$$Tf\colon T\,Base^T(\alpha) \to TPC.$$

We claim that $\Psi := Tf(\alpha)$ is an element of $SRD(\Gamma)$ and that $\alpha \mathrel{\overline{T}{\leq}} T\bigwedge(\Psi)$. For the first claim, since $\Psi \in TPC$, all we need to show is that $\Gamma \subseteq \lambda^T(\Psi)$, i.e. that for all $\gamma \in \Gamma$, $\gamma \mathrel{\overline{T}{\in}} \Psi$. So suppose that $\gamma \in \Gamma$; then by assumption, $\alpha \mathrel{\overline{T}{\leq}} \gamma$, so $Base^T(\alpha) \mathrel{\overline{P}{\leq}} Base^T(\gamma)$. It follows from the definition of f that for all $b \in Base^T(\gamma)$, and all $a \in Base^T(\alpha)$, if $a \leq b$ then $b \in f(a)$. It follows by Fact 2.6 that

$$\forall\delta \in T\,Base^T(\alpha),\, \forall\beta \in T\,Base^T(\gamma),\, \delta \mathrel{\overline{T}{\leq}} \beta \Rightarrow \beta \mathrel{\overline{T}{\in}} Tf(\delta).$$

So in particular, since $\alpha \in T\,Base^T(\alpha)$, $\gamma \in T\,Base^T(\gamma)$ and $\alpha\ \overline{T}{\leq}\ \gamma$, we see that $\gamma\ \overline{T}{\in}\ Tf(\alpha) = \Psi$. Since $\gamma \in \Gamma$ was arbitrary, it follows that $\Gamma \subseteq \lambda^T(\Psi)$. Consequently, $\Psi \in SRD(\Gamma)$, as we wanted to show.

For the second claim, i.e. that $\alpha\ \overline{T}{\leq}\ T\bigwedge(\Psi)$, it suffices to observe that $a \leq \bigwedge f(a)$ for all $a \in Base^T(\alpha)$, so by Fact 2.6,

$$\forall \delta \in T\,Base^T(\alpha),\ \delta\ \overline{T}{\leq}\ T\textstyle\bigwedge \circ Tf(\delta).$$

Since $\alpha \in T\,Base^T(\alpha)$ and $\Psi = Tf(\alpha)$, we get that $\alpha\ \overline{T}{\leq}\ T\bigwedge \circ Tf(\alpha) = T\bigwedge(\Psi)$. □

Corollary 3.6. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor and let $\mathbb{L}$ be a frame. Then*

$$V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL \mid (\nabla 1), (\nabla 2'), (\nabla 3)\rangle,$$

where the relations are as follows:

$$\begin{array}{lll} (\nabla 1) & \nabla\alpha \leq \nabla\beta, & (\alpha\ \overline{T}{\leq}\ \beta) \\ (\nabla 2') & \bigwedge_{\gamma\in\Gamma}\nabla\gamma \leq \bigvee\{\nabla\alpha \mid \forall\gamma \in \Gamma,\ \alpha\ \overline{T}{\leq}\ \gamma\}, & (\Gamma \in P_\omega TL) \\ (\nabla 3) & \nabla(T\bigvee)\Phi \leq \bigvee\{\nabla\beta \mid \beta\ \overline{T}{\in}\ \Phi\}, & (\Phi \in TPL) \end{array}$$

Proof. Observe that the only difference between $\mathrm{Fr}\langle TL \mid (\nabla 1), (\nabla 2'), (\nabla 3)\rangle$ and the original definition of $V_T\mathbb{L}$ is that we replaced $(\nabla 2)$,

$$(\nabla 2) \quad \textstyle\bigwedge_{\alpha\in\Gamma}\nabla\alpha \leq \bigvee\{\nabla(T\bigwedge)\Psi \mid \Psi \in SRD(\Gamma)\}, \quad (\Gamma \in P_\omega TL)$$

with $(\nabla 2')$. The equivalence of these two relations is an immediate corollary of Lemma 3.5: take any $\Gamma \in TP_\omega L$, then

$$\begin{array}{ll} \bigvee\{\nabla T\bigwedge(\Psi) \mid \Psi \in SRD(\Gamma)\} & \\ = \bigvee\{\nabla\alpha \mid \exists\Psi \in SRD(\Gamma),\ \alpha\ \overline{T}{\leq}\ \nabla T\bigwedge(\Psi)\} & \text{by order theory and } (\nabla 1), \\ = \bigvee\{\nabla\alpha \mid \forall\gamma \in \Gamma,\ \alpha\ \overline{T}{\leq}\ \gamma\} & \text{by Lemma 3.5.} \end{array}$$

It follows that $V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL \mid (\nabla 1), (\nabla 2'), (\nabla 3)\rangle$. □

Remark 3.7. We will see later that both axioms $(\nabla 2)$ and $(\nabla 2')$ are equally useful. It seems that $(\nabla 2')$ has not been studied before in the literature on coalgebraic modal logic via the ∇-modality (Palmigiano & Venema, 2007; Bílková et al., 2008; Kissig & Venema, 2009; Kupke et al., 2010).

3.2. **Basic properties of the T-powerlocale.** In this subsection we make some technical observations about slim redistributions and about the structure of the T-powerlocale. We start with two facts on slim redistributions.

Lemma 3.8. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor. Then $SRD(\varnothing) = T\{\varnothing\}$.*

Proof. If Φ is a slim redistribution of the empty set, then by definition $\Phi \in TP_\omega(\varnothing) = T\{\varnothing\}$. Conversely, any $\Phi \in T\{\varnothing\}$ satisfies the condition that $\varnothing \subseteq \lambda^T(\Phi)$, and so $\Phi \in SRD(\varnothing)$. □

The following Lemma plays an essential role when defining V_T on frame homomorphisms, rather than just on frames. It is of crucial use when showing that if $f\colon \mathbb{L} \to \mathbb{M}$ is a frame homomorphism, then $V_Tf\colon V_T\mathbb{L} \to V_T\mathbb{M}$ preserves conjunctions, as we will see in §3.4.

Lemma 3.9. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor, let X, Y be sets and let $f\colon X \to Y$ be a function; let $\Gamma \in P_\omega TX$. Then the restriction of $TP_\omega f\colon TP_\omega X \to TP_\omega Y$ to $SRD(\Gamma)$ is a surjection onto $SRD(P_\omega Tf\Gamma)$.*

Proof. Let X, Y, f and Γ be as in the statement of the Lemma, and abbreviate $\Gamma' := (P_\omega Tf)\Gamma$, $C := \bigcup_{\gamma\in\Gamma} Base^T(\gamma)$ and $C' := \bigcup_{\gamma'\in\Gamma'} Base^T(\gamma')$. Then an easy calculation shows that

$$\begin{aligned} C' &= \bigcup_{\gamma\in\Gamma} Base^T(Tf)(\gamma) && \text{(definition of } \Gamma'\text{)} \\ &= \bigcup_{\gamma\in\Gamma} (Pf)\, Base^T(\gamma) && (Base^T \text{ is natural transformation)} \\ &= (Pf)(C) && \text{(elementary set theory)} \end{aligned}$$

We will first show that $TP_\omega f$ maps slim redistributions of Γ to slim redistributions of Γ'. For that purpose, take an arbitrary element $\Phi \in SRD(\Gamma)$, and write $\Phi' := (TP_\omega f)\Phi$. We claim that $\Phi' \in SRD(\Gamma')$, and first show that

$$\Phi' \in TP_\omega C', \tag{1}$$

or equivalently, that $Base^T\,\Phi' \subseteq P_\omega C'$. To prove this inclusion, take an arbitrary set $A' \in Base^T(\Phi')$. Since by Fact 2.8, $Base^T(\Phi') = (P_\omega P_\omega f)(Base^T(\Phi)$, this means that A' must be of the form $(P_\omega f)(A)$ for some $A \in Base^T(\Phi)$. In particular, A' must be a subset of $(P_\omega f)(\bigcup Base^T(\Phi))$. Also, because Φ is a slim redistribution of Γ, by definition we have $Base^T(\Phi) \subseteq P_\omega C$, and so $\bigcup Base^T(\Phi) \subseteq \bigcup C$. From this it follows that $A' \subseteq (Pf)(\bigcup Base^T(\Phi)) \subseteq (Pf)(\bigcup C) = C'$, as required.

Second, we claim that

$$\Gamma' \subseteq \lambda^T(\Phi'). \tag{2}$$

To prove this, take an arbitrary element of Γ', say, $(Tf)\gamma$ for some $\gamma \in \Gamma$. We have $\gamma\ \overline{T}{\in}\ \Phi$ by the assumption that $\Phi \in SRD(\Gamma)$. But then, since $a \in A$ implies $fa \in (P_\omega f)A$ for any $a \in C$ and $A \subseteq C$, it follows by Lemma 2.7 that $\gamma' = (Tf)\gamma\ \overline{T}{\in}\ (TP_\omega f)(\Phi) = \Phi'$. This means that γ' is a lifted member of Φ', as required.

Clearly, the claims (1) and (2) above suffice to prove that $\Phi' \in SRD(\Gamma')$, which means that indeed, $TP_\omega f$ maps slim redistributions of Γ to slim redistributions of Γ'.

Thus it is left to prove that every slim redistribution of Γ' is of the form $(TP_\omega f)\Phi$ for some slim redistribution Φ of Γ. Take an arbitrary $\Phi' \in SRD(\Gamma')$, and recall that $\breve{P}$ denotes the contravariant power set functor. Restrict f to the map $f^-\colon C \to C'$, which means that $\breve{P}f^-\colon P_\omega C' \to P_\omega C$. It follows that $T\breve{P}f^-\colon TP_\omega C' \to TP_\omega C$, so that we may define $\Phi := (T\breve{P}f^-)\Phi'$, and obtain $\Phi \in TP_\omega C$. Hence, in order to prove that

$$\Phi \in SRD(\Gamma), \tag{3}$$

it suffices to show that $\Gamma \subseteq \lambda^T(\Phi)$. But this is an immediate consequence of the fact that λ^T is a distributive law of T over $\breve{P}$ (Fact 2.9), since for an arbitrary $\gamma \in \Gamma$ we may reason as follows. From $\gamma \in \Gamma$ it follows by definition of Γ' that $(Tf^-)(\gamma) = (Tf)(\gamma)$ belongs to Γ'. Since $\Gamma' \subseteq \lambda^T_Y(\Phi)$ by assumption, by definition of $\breve{P}$ we find that $\gamma \in (\breve{P}Tf)\lambda^T_Y(\Psi)$. But by $\lambda^T\colon T\breve{P} \to \breve{P}T$ we know that $(\breve{P}Tf)\lambda^T_Y(\Psi) = \lambda^T_X(T\breve{P}f)(\Psi) = \lambda^T_X(\Phi)$. Thus we find $\gamma \in \lambda^T(\Phi)$, as required.

Finally, observe that $f^-\colon C \to C'$ is surjective, so that it follows by properties of the co- and contravariant power set functors that $P_\omega f^- \circ \breve{P}f^- = id_{P_\omega C'}$. From this it is immediate by functoriality of T that

$$\Phi' = (TP_\omega f^- \circ T\breve{P}f^-)\Phi' = (TP_\omega f^-)\Phi = (TP_\omega f)\Phi.$$

This finishes the proof of the Lemma. □

In the following lemma we gather some basic observations on the frame structure of the T-powerlocale. These facts generalize results from (Kupke et al., 2010) to our geometrical setting.

Lemma 3.10. *Let T be a standard, finitary, weak pullback-preserving functor and let $\mathbb{L}$ be a frame.*

(1) *If $\alpha \in TL$ is such that $0_{\mathbb{L}} \in Base^T(\alpha)$, then $\nabla\alpha = 0_{V_T\mathbb{L}}$.*

(2) *If* $A \subseteq L$ *is such that* $a \wedge b = 0_{\mathbb{L}}$ *for all* $a \neq b$ *in* A, *then* $\nabla\alpha \wedge \nabla\beta = 0_{V_T\mathbb{L}}$ *for all* $\alpha \neq \beta$ *in* TA.
(3) *If there is no relation* R *such that* $\alpha\, \overline{T}R\, \beta$, *then* $\nabla\alpha \wedge \nabla\beta = 0_{V_T\mathbb{L}}$.
(4) $1_{V_T\mathbb{L}} = \bigvee\{\nabla\gamma \mid \gamma \in T\{1_{\mathbb{L}}\}\}$.
(5) *For any* $A \subseteq L$ *such that* $1_{\mathbb{L}} = \bigvee A$, *we have* $1_{V_T\mathbb{L}} = \bigvee\{\nabla\alpha \mid \alpha \in TA\}$.

Proof. For part 1, let $\alpha \in TL$ be such that $0_{\mathbb{L}} \in \mathit{Base}^T(\alpha)$. Consider the map $f\colon L \to PL$ given by

$$f(a) := \begin{cases} \emptyset & \text{if } a = 0_{\mathbb{L}}, \\ \{a\} & \text{if } a > 0_{\mathbb{L}}. \end{cases}$$

Then $id_L = \bigvee \circ f$, so that $id_{TL} = (T\bigvee) \circ (Tf)$ by functoriality of T. In particular, we obtain that $\alpha = (T\bigvee)(Tf)(\alpha)$, so that we may calculate

$$\begin{aligned} \nabla\alpha &= \bigvee\left\{\nabla\beta \mid \beta\, \overline{T}{\in}\, (Tf)(\alpha)\right\} && \text{(axiom } \nabla 2\text{)} \\ &\leq \bigvee\left\{\nabla\beta \mid \mathit{Base}^T(\beta)\, \overline{P}{\in}\, \mathit{Base}^T((Tf)(\alpha))\right\} && \text{(Fact 2.8(2))} \\ &= \bigvee\emptyset && (\dagger) \\ &= 0_{V_T\mathbb{L}} \end{aligned}$$

In order to justify the remaining step (†) in this calculation, observe that it follows from the naturality of Base^T (Fact 2.8(1)) that

$$\mathit{Base}^T((Tf)(\alpha)) = (Pf)(\mathit{Base}^T(\alpha)),$$

and so by the assumption that $0_{\mathbb{L}} \in \mathit{Base}^T(\alpha)$ we obtain $\emptyset \in \mathit{Base}^T((Tf)(\alpha))$. Now suppose for contradiction that there is some $B \subseteq L$ such that $B\, \overline{P}{\in}\, \mathit{Base}^T((Tf)(\alpha))$. Then by definition of $\overline{P}$ there is a $b \in B$ such that $b \in \emptyset$, which provides the desired contradiction. This proves (†), and finishes the proof of part 1.

For part 2, let $A \subseteq L$ be such that $a \wedge b = 0_{\mathbb{L}}$ for all $a \neq b$ in A, and take two distinct elements $\alpha, \beta \in TA$. In order to prove that $\nabla\alpha \wedge \nabla\beta = 0_{V_T\mathbb{L}}$, it suffices by axiom ($\nabla$2) to show that

$$\nabla(T\textstyle\bigwedge)(\Phi) = 0_{V_T\mathbb{L}}, \text{ for all } \Phi \in \mathit{SRD}\{\alpha, \beta\}. \tag{4}$$

Take an arbitrary slim redistribution Φ of $\{\alpha, \beta\}$, then by Fact 2.11, $\mathit{Base}^T(\Phi)$ contains a set $A_0 \subseteq_\omega A$ of size > 1. Define the map $d\colon\ \mathit{Base}^T(\Phi) \to P_\omega(A) \cup \{\{1_{\mathbb{L}}\}\}$ by putting:

$$d(B) := \begin{cases} \emptyset & \text{if } |B| > 1, \\ B & \text{if } |B| = 1, \\ \{1_{\mathbb{L}}\} & \text{if } |B| = 0. \end{cases}$$

It is straightforward to verify from the assumptions on A and the definition of d, that $\bigwedge B \leq \bigvee d(B)$, for each $B \in \mathit{Base}^T(\Phi)$. Hence it follows by Fact 2.6 that $(T\bigwedge)(\Phi)\, \overline{T}{\leq}\, (T\bigvee)(Td)(\Phi)$, so that by axiom ($\nabla$1) we may conclude that

$$\nabla(T\textstyle\bigwedge)(\Phi) \leq \nabla(T\textstyle\bigvee)(Td)(\Phi) \tag{5}$$

Finally, it follows from the naturality of Base^T (Fact 2.8(1)) that $\mathit{Base}^T(Td)(\Phi) = (Pd)(\mathit{Base}^T(\Phi))$. Consequently, for the set $A_0 \in \mathit{Base}^T(\Phi)$ satisfying $|A_0| > 1$, we find $\emptyset = d(A_0) \in \mathit{Base}^T(Td)(\Phi)$, and then $0_{\mathbb{L}} = \bigvee\varnothing \in (P\bigvee)\,\mathit{Base}^T(Td)(\Phi) = \mathit{Base}^T(T\bigvee)(Td)(\Phi)$. Thus by part (1) of this lemma it follows that

$$\nabla(T\textstyle\bigvee)(Td)(\Phi) = 0_{V_T\mathbb{L}}. \tag{6}$$

This finishes the proof of part 2, since (4) is immediate on the basis of (5) and (6).

In order to prove part 3, suppose that $\alpha, \beta \in TL$ are not linked by any lifted relation. Consider the (unique) map

$$f\colon L \to \{1\},$$

and define $\alpha' := (Tf)\alpha$, $\beta' := (Tf)(\beta)$. Suppose for contradiction that $\alpha' = \beta'$. Then we would find $\alpha\, \overline{T}((Gr\, f)^{\smile}\,;\, Gr\, f)\, \beta$, contradicting the assumption on α and β. It follows that α' and β are distinct, and so by part (2) of this lemma (with $A = \{1_{\mathbb{L}}\}$), we may infer that $\nabla\alpha' \wedge \nabla\beta' = 0_{V_T\mathbb{L}}$. This means that we are done, since it follows from $Gr\, f \subseteq\, \leq$ and the definitions of α', β', that $\alpha\, \overline{T}{\leq}\, \alpha'$ and $\beta\, \overline{T}{\leq}\, \beta'$, and from this we obtain by (∇1) that

$$\nabla\alpha \wedge \nabla\beta \leq \nabla\alpha' \wedge \nabla\beta' \leq 0_{V_T\mathbb{L}}.$$

For part 4, we reason as follows:

$$\begin{aligned}
1_{V_T\mathbb{L}} &= \bigvee\{\nabla(T\textstyle\bigwedge)(\Phi) \mid \Phi \in SRD(\emptyset)\}, && \text{(axiom } (\nabla 2) \text{ with } A = \emptyset) \\
&= \bigvee\{\nabla(T\textstyle\bigwedge)(\Phi) \mid \Phi \in T\{\emptyset\}\} && \text{(Fact 3.8)} \\
&= \bigvee\{\nabla\gamma \mid \gamma \in T\{1_{\mathbb{L}}\}\} && (\ddagger)
\end{aligned}$$

where the last step ($\ddagger$) is justified by the observation that, since the map $\bigwedge\colon P_\omega L \to L$ restricts to a bijection $\bigwedge\colon \{\emptyset\} \to \{1_{\mathbb{L}}\}$, its lifting restricts to a bijection $T\bigwedge\colon T\{\emptyset\} \to T\{1_{\mathbb{L}}\}$.

Finally, we turn to the proof of part 5. Let $A \subseteq L$ be such that $1_{\mathbb{L}} = \bigvee A$, and consider an arbitrary element $\Phi \in T\{A\}$. We claim that

$$\lambda^T(\Phi) \subseteq TA. \tag{7}$$

To see this, take an arbitrary lifted element α of Φ. It follows from $\alpha\, \overline{T}{\in}\, \Phi$ that $Base^T(\alpha)\, \overline{P}{\in}\, Base^T(\Phi)$. In particular, each $a \in Base^T(\alpha)$ must belong to some $B \in Base^T(\Phi) \subseteq \{A\}$. In other words, $Base^T(\alpha) \subseteq A$, which is equivalent to saying that $\alpha \in TA$. This proves (7).

By (7) and axiom (∇3) we obtain

$$\nabla(T\textstyle\bigvee)(\Phi) \leq \bigvee\{\nabla\alpha \mid \alpha \in TA\}. \tag{8}$$

Now we reason as follows:

$$\begin{aligned}
1_{V_T\mathbb{L}} &= \bigvee\{\nabla\alpha \mid \alpha \in T\{1_{\mathbb{L}}\}\} && \text{(part 4)} \\
&= \bigvee\{\nabla(T\textstyle\bigvee)(\Phi) \mid \Phi \in T\{A\}\} && (*) \\
&\leq \bigvee\{\nabla\alpha \mid \alpha \in TA\}, && (8)
\end{aligned}$$

To justify the second step ($*$), observe that if we restrict the map $\bigvee\colon PL \to L$ to the bijection $\bigvee\colon \{A\} \to \{1_{\mathbb{L}}\}$, as its lifting we obtain a bijection $T\bigvee\colon T\{A\} \to T\{1_{\mathbb{L}}\}$. □

3.3. **Two examples of the T-powerlocale construction.** In this subsection we will discuss two examples of T-powerlocales. First, we discuss the somewhat trivial example of the Id-powerlocale. After that, we will discuss the defining example of T-powerlocales, namely the P_ω-powerlocale, which is isomorphic to the classical Vietoris powerlocale.

Example 3.11. Let $\mathrm{Id}\colon \mathsf{Set} \to \mathsf{Set}$ be the identity functor on the category of sets. Then for all frames $\mathbb{L}$, $V_{\mathrm{Id}}\mathbb{L} \simeq \mathbb{L}$.

First recall from Example 2.4 that for any relation $R \subseteq X \times Y$, $\overline{\mathrm{Id}}R = R$. Moreover, if $A \in \mathrm{Id}\, P_\omega L = P_\omega L$, then it is straightforward to verify that

$$\begin{aligned}
SRD(A) &= \{\Psi \in P_\omega(\textstyle\bigcup_{c\in A}\{c\}) \mid \forall c \in A, c \in \Psi\} \\
&= \{A\}.
\end{aligned}$$

Consequently, the ∇-relations reduce to the following in case $T = \mathrm{Id}$:

$$\begin{array}{lll}
(\nabla 1) & \nabla a \leq \nabla b, & (a \leq b) \\
(\nabla 2) & \bigwedge_{a\in A}\nabla a \leq \nabla\bigwedge A, & (A \in P_\omega L) \\
(\nabla 3) & \nabla\bigvee A \leq \bigvee\{\nabla b \mid b \in A\}. & (A \in PL)
\end{array}$$

The identity $id_L \colon L \to L$ obviously satisfies (∇1), (∇2) and (∇3). Moreover if we have a frame $\mathbb{M}$ and a function $f \colon L \to M$ which is compatible with (∇1), (∇2) and (∇3), then it is easy to see that f is in fact a frame homomorphism $\mathbb{L} \to \mathbb{M}$. By the universal property of frame presentations, it follows that $V_{\mathrm{Id}}\mathbb{L} \simeq \mathbb{L}$.

We now turn to the P_ω-powerlocale. Recall from Example 2.2 that $P_\omega \colon \mathsf{Set} \to \mathsf{Set}$, the covariant finite power set functor, is indeed standard, weak pullback-preserving and finitary. We will now show that the P_ω-powerlocale is the Vietoris powerlocale. The equivalence of the ∇ axioms and the $\Box$, $\Diamond$ axioms on distributive lattices is already known from the work of Palmigiano & Venema (2007); what is different here is that we consider infinite joins rather than only finite joins.

We will use the presentation using (∇1), ($\nabla 2'$) and (∇3) as our point of departure. Recall that for all $\alpha, \beta \in P_\omega L$,

$$\begin{aligned}
&\alpha \le_L \beta \text{ if } \alpha \subseteq {\downarrow}\beta,\\
&\alpha \le_U \beta \text{ if } {\uparrow}\alpha \supseteq \beta,\\
&\alpha \le_C \beta \text{ if } \alpha \le_L \beta \text{ and } \alpha \le_U \beta.
\end{aligned}$$

By Example 3.2, two of the relations presenting $V_{P_\omega}\mathbb{L}$ thus become

$$\begin{array}{ll}
(\nabla 2') & \bigwedge_{\gamma\in\Gamma} \nabla\gamma \le \bigvee \{\nabla\alpha \mid \forall\gamma \in \Gamma,\, \alpha \le_C \gamma\}\\
(\nabla 3) & \nabla \{\bigvee \alpha \mid \alpha \in \Phi\} \le \bigvee \{\nabla\beta \mid \beta \in P_\omega \left(\bigcup \Phi\right) \text{ and } \forall\alpha \in \Phi,\, \alpha \between \beta\}
\end{array}$$

Lemma 3.12. *We consider the presentation of $V_{P_\omega}\mathbb{L}$.*

(1) *In the presence of* (∇1)*, the relation* ($\nabla 2'$) *can be replaced by*

$$\begin{array}{ll}
(\nabla 2.0) & 1 \le \bigvee \{\nabla\beta \mid \beta \in P_\omega L\}\\
(\nabla 2.2) & \nabla\gamma_1 \wedge \nabla\gamma_2 \le \bigvee \{\nabla\beta \mid \beta \le_C \gamma_1, \beta \le_C \gamma_2\}
\end{array}$$

(2) *In the presence of* (∇1) *and* (∇2) *(or its equivalent formulations), the relation* (∇3) *can be replaced by*

$$\begin{array}{ll}
(\nabla 3.{\uparrow}) & \nabla \left(\gamma \cup \{\bigvee^{\uparrow} S\}\right) \le \bigvee^{\uparrow} \{\nabla (\gamma \cup \{a\}) \mid a \in S\} \quad \textit{(S directed)}\\
(\nabla 3.0) & \nabla (\gamma \cup \{0\}) \le 0\\
(\nabla 3.2) & \nabla (\gamma \cup \{a_1 \vee a_2\}) \le \nabla (\gamma \cup \{a_1\}) \vee \nabla (\gamma \cup \{a_2\}) \vee \nabla (\gamma \cup \{a_1, a_2\})
\end{array}$$

Proof. (1) (∇2.0) and (∇2.2) are special cases of ($\nabla 2'$), when Γ is empty or a doubleton. To show that they imply ($\nabla 2'$) is an induction on the number of elements needed to enumerate the finite set Γ.

(2) Each of the replacement relations is a special case of (∇3) in which all except one of the elements of Φ are singletons. We now show that they are sufficient to imply (∇3). First, we show for any *finite* S that

$$\nabla \left(\gamma \cup \{\bigvee S\}\right) \le \bigvee \{\nabla (\gamma \cup \alpha) \mid \emptyset \ne \alpha \in P_\omega S\}.$$

We use induction on the length of a finite enumeration of S. The base case, S empty, is (∇3.0). Now suppose $S = \{a\} \cup S'$. Then

$$\begin{aligned}
&\nabla \left(\gamma \cup \{\bigvee S\}\right)\\
&= \nabla \left(\gamma \cup \{a \vee \bigvee S'\}\right)\\
&\le \nabla (\gamma \cup \{a\}) \vee \nabla \left(\gamma \cup \{\bigvee S'\}\right) \vee \nabla \left((\gamma \cup \{a\}) \cup \{\bigvee S'\}\right) && \text{(by } (\nabla 3.2))\\
&\le \nabla (\gamma \cup \{a\}) \vee \bigvee \{\nabla (\gamma \cup \alpha') \mid \emptyset \ne \alpha' \in P_\omega S'\}\\
&\qquad \vee \bigvee \{\nabla (\gamma \cup \{a\} \cup \alpha') \mid \emptyset \ne \alpha' \in P_\omega S'\} && \text{(by induction)}\\
&= \bigvee \{\nabla (\gamma \cup \alpha) \mid \emptyset \ne \alpha \in P_\omega S\}.
\end{aligned}$$

Now we can use (∇3.$\uparrow$) to relax the finiteness condition on S, since for an arbitrary S we have

$$\begin{aligned}\nabla\left(\gamma \cup \{\bigvee S\}\right) &= \nabla\left(\gamma \cup \left\{\bigvee\nolimits^{\uparrow}\left\{\bigvee S_0 \mid S_0 \in P_\omega S\right\}\right\}\right)\\ &\leq \bigvee\nolimits^{\uparrow}\left\{\nabla\left(\gamma \cup \left\{\bigvee S_0\right\}\right) \mid S_0 \in P_\omega S\right\}.\end{aligned}$$

Finally, we can use induction on the length of a finite enumeration of Φ to deduce (∇3). More precisely, one shows by induction on n that

$$\begin{aligned}&\nabla\left(\gamma \cup \{\bigvee S_1, \ldots, \bigvee S_n\}\right)\\ &\leq \bigvee\left\{\nabla\left(\gamma \cup \alpha\right) \mid \emptyset \neq \alpha \in P_\omega\left(\bigcup_{i=1}^{n} S_i\right) \text{ and } \forall i,\, \alpha \between S_i\right\}.\end{aligned}$$

$\square$

Remark 3.13. Relation (∇2.0) can be weakened even further, to

$$1 \leq \nabla\emptyset \vee \nabla\{1\}.$$

For if β is non-empty then $\beta \leq_C \{1\}$. From (∇2.2) we can also deduce that $\nabla\emptyset \wedge \nabla\{1\} = 0$, giving that $\nabla\emptyset$ and $\nabla\{1\}$ are clopen complements.

Lemma 3.14. *In $V\mathbb{L}$ we have, for any $S \subseteq \mathbb{L}$,*

$$\Box\left(\bigvee S\right) = \bigvee\left\{\Box\left(\bigvee \alpha\right) \wedge \bigwedge_{a \in \alpha} \Diamond a \mid \alpha \in P_\omega S\right\}.$$

Proof. $\geq$ is immediate. For $\leq$, first note that since $\bigvee S$ is a directed join $\bigvee^{\uparrow}_{\alpha \in P_\omega S} \bigvee \alpha$, we have $\Box\left(\bigvee S\right) \leq \bigvee^{\uparrow}_{\alpha \in P_\omega S} \Box\left(\bigvee \alpha\right)$ and thus we reduce to the case where S is finite. We show that for every $\alpha, \beta \in P_\omega S$ we have

$$\Box\left(\bigvee \alpha \vee \bigvee \beta\right) \wedge \bigwedge_{a \in \alpha} \Diamond a \leq \text{ RHS in statement,}$$

after which the result follows by taking $\beta = S$ and $\alpha = \emptyset$. We use P_ω-induction on β, effectively an induction on the length of an enumeration of its elements. The base case, $\beta = \emptyset$, is trivial. For the induction step, suppose $\beta = \beta' \cup \{b\}$. Then

$$\begin{aligned}&\Box\left(\bigvee \alpha \vee \bigvee \beta\right) \wedge \bigwedge_{a \in \alpha} \Diamond a\\ &= \Box\left(\bigvee \alpha \vee b \vee \bigvee \beta'\right) \wedge \bigwedge_{a \in \alpha} \Diamond a\\ &= \Box\left(\bigvee \alpha \vee b \vee \bigvee \beta'\right) \wedge \bigwedge_{a \in \alpha} \Diamond a \wedge \left(\Box\left(\bigvee \alpha \vee \bigvee \beta'\right) \vee \Diamond b\right)\\ &= \left(\Box\left(\bigvee \alpha \vee \bigvee \beta'\right) \wedge \bigwedge_{a \in \alpha} \Diamond a\right) \vee \left(\Box\left(\bigvee (\alpha \cup \{b\}) \vee \bigvee \beta'\right) \wedge \bigwedge_{a \in \alpha \cup \{b\}} \Diamond a\right)\\ &\leq \text{ RHS, by induction.}\end{aligned}$$

$\square$

Theorem 3.15. *Let $\mathbb{L}$ be a frame. Then $V\mathbb{L} \cong V_{P_\omega}\mathbb{L}$.*

Proof. First we define a frame homomorphism $\varphi\colon V_{P_\omega}\mathbb{L} \to V\mathbb{L}$ by $\varphi(\nabla\alpha) = \Box\left(\bigvee\alpha\right) \wedge \bigwedge_{a\in\alpha}\Diamond a$. We must check that this respects the relations. For (∇1), suppose $\alpha \leq_C \beta$. From $\alpha \leq_U \beta$ and $\alpha \leq_L \beta$ we get $\bigwedge_{a\in\alpha}\Diamond a \leq \bigwedge_{b\in\beta}\Diamond b$ and $\bigvee\alpha \leq \bigvee\beta$, giving $\varphi(\nabla\alpha) \leq \varphi(\nabla\beta)$.

For (∇2.0), we have $1 = \Box(0\vee 1) = \Box 0 \vee (\Box 1 \wedge \Diamond 1) = \varphi\left(\nabla\emptyset\right) \vee \varphi\left(\nabla\left\{1\right\}\right)$.

For (∇2.2), $\varphi\left(\nabla\gamma_1\right) \wedge \varphi\left(\nabla\gamma_2\right)$ is

$$\begin{aligned}
&\Box\left(\bigvee\gamma_1\right) \wedge \bigwedge\nolimits_{c\in\gamma_1}\Diamond c \wedge \Box\left(\bigvee\gamma_2\right) \wedge \bigwedge\nolimits_{c'\in\gamma_2}\Diamond c' \\
&= \Box\left(\bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \wedge \bigwedge\nolimits_{c\in\gamma_1}\Diamond c \wedge \bigwedge\nolimits_{c'\in\gamma_2}\Diamond c' \\
&= \Box\left(\bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \wedge \bigwedge\nolimits_{c\in\gamma_1}\Diamond\left(c \wedge \bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \wedge \bigwedge\nolimits_{c'\in\gamma_2}\Diamond\left(c' \wedge \bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \\
&= \Box\left(\bigvee\gamma_1 \wedge \bigvee\gamma_2\right) \wedge \bigwedge\nolimits_{c\in\gamma_1}\Diamond\left(c \wedge \bigvee\gamma_2\right) \wedge \bigwedge\nolimits_{c'\in\gamma_2}\Diamond\left(c' \wedge \bigvee\gamma_1\right) \\
&= \Box\left(\bigvee\nolimits_{c\in\gamma_1}\bigvee\nolimits_{c'\in\gamma_2} c \wedge c'\right) \wedge \bigwedge\nolimits_{c\in\gamma_1}\bigvee\nolimits_{c'\in\gamma_2}\Diamond\left(c \wedge c'\right) \wedge \bigwedge\nolimits_{c'\in\gamma_2}\bigvee\nolimits_{c\in\gamma_1}\Diamond\left(c \wedge c'\right)
\end{aligned}$$

Redistributing the disjunctions of the $\Diamond$s, we find that each resulting disjunct is of the form

$$\Box\left(\bigvee\nolimits_{c\in\gamma_1}\bigvee\nolimits_{c'\in\gamma_2} c \wedge c'\right) \wedge \bigwedge\nolimits_{cRc'}\Diamond\left(c \wedge c'\right)$$

for some $R \in P_\omega\left(\gamma_1 \times \gamma_2\right)$ such that $\gamma_1 \,\overline{P_\omega}R\, \gamma_2$. Note that for any such R if we define $\beta_R = \{c \wedge c' \mid cRc'\}$ then we have $\beta_R \leq_C \gamma_i$ $(i = 1, 2)$. Now by Lemma 3.14 we see

$$\begin{aligned}
&\Box\left(\bigvee\nolimits_{c\in\gamma_1}\bigvee\nolimits_{c'\in\gamma_2} c \wedge c'\right) \wedge \bigwedge\nolimits_{cRc'}\Diamond\left(c \wedge c'\right) \\
&\leq \bigvee\left\{\Box\left(\bigvee\nolimits_{cR'c'} c \wedge c'\right) \wedge \bigwedge\nolimits_{c(R\cup R')c'}\Diamond\left(c \wedge c'\right) \mid R' \in P_\omega\left(\gamma_1 \times \gamma_2\right)\right\} \\
&\leq \bigvee\left\{\Box\left(\bigvee\nolimits_{cR'c'} c \wedge c'\right) \wedge \bigwedge\nolimits_{cR'c'}\Diamond\left(c \wedge c'\right) \mid R \subseteq R' \in P_\omega\left(\gamma_1 \times \gamma_2\right)\right\} \\
&= \bigvee\left\{\varphi\left(\nabla\beta_{R'}\right) \mid R \subseteq R' \in P_\omega\left(\gamma_1 \times \gamma_2\right)\right\}
\end{aligned}$$

and the result follows.

For (∇3.$\uparrow$): the LHS is

$$\begin{aligned}
&\Box\left(\bigvee\gamma \vee \bigvee\nolimits^{\uparrow} S\right) \wedge \bigwedge\nolimits_{c\in\gamma}\Diamond c \wedge \Diamond\left(\bigvee\nolimits^{\uparrow} S\right) \\
&= \bigvee\nolimits^{\uparrow}\left\{\Box\left(\bigvee\gamma \vee a\right) \mid a \in S\right\} \wedge \bigvee\nolimits^{\uparrow}\left\{\bigwedge\nolimits_{c\in\gamma}\Diamond c \wedge \Diamond a \mid a \in S\right\} \\
&= \bigvee\nolimits^{\uparrow}\left\{\Box\left(\bigvee\gamma \vee a\right) \wedge \bigwedge\nolimits_{c\in\gamma}\Diamond c \wedge \Diamond a \mid a \in S\right\}
\end{aligned}$$

which is the RHS.

For (∇3.0): the LHS is

$$\Box\left(\bigvee\gamma \vee 0\right) \wedge \bigwedge\nolimits_{c\in\gamma}\Diamond c \wedge \Diamond 0 = 0.$$

For (∇3.2): the LHS is

$$\begin{aligned}
&\Box\left(\bigvee\gamma \vee a_1 \vee a_2\right) \wedge \bigwedge\nolimits_{c\in\gamma}\Diamond c \wedge \Diamond\left(a_1 \vee a_2\right) \\
&= \bigvee_{i=1}^{2}\bigvee\left\{\Box\left(\bigvee\beta\right) \wedge \bigwedge\nolimits_{c\in\beta\cup\gamma\cup\{a_i\}}\Diamond c \mid \beta \in P_\omega\left(\gamma \cup \{a_1, a_2\}\right)\right\} \\
&\leq \bigvee_{i=1}^{2}\bigvee\left\{\varphi\left(\nabla\left(\beta \cup \gamma \cup \{a_i\}\right)\right) \mid \beta \in P_\omega\left(\gamma \cup \{a_1, a_2\}\right)\right\} \\
&= \bigvee_{i=1}^{2}\bigvee\left\{\varphi\left(\nabla\beta\right) \mid \gamma \cup \{a_i\} \subseteq \beta \in P_\omega\left(\gamma \cup \{a_1, a_2\}\right)\right\} \\
&= \varphi\left(\nabla\left(\gamma \cup \{a_1\}\right)\right) \vee \varphi\left(\nabla\left(\gamma \cup \{a_2\}\right)\right) \vee \varphi\left(\nabla\left(\gamma \cup \{a_1, a_2\}\right)\right).
\end{aligned}$$

Next, we define the frame homomorphism $\psi\colon V\mathbb{L} \to V_{P_\omega}\mathbb{L}$ by

$$\begin{aligned}\psi(\Box a) &= \bigvee\{\nabla\alpha \mid \alpha \leq_L \{a\}\} = \nabla\emptyset \vee \nabla\{a\}\\ \psi(\Diamond a) &= \bigvee\{\nabla\alpha \mid \alpha \leq_U \{a\}\} = \bigvee\{\nabla(\beta \cup \{a\}) \mid \beta \in P_\omega L\}.\end{aligned}$$

(Observe that the expression for $\psi(\Diamond a)$ could be simplified even further to $\nabla\{1, a\}$.) We check the relations. First, it is clear that ψ respects monotonicity of $\Box$ and $\Diamond$.

$\Box$ preserves directed joins:

$$\psi\left(\Box\left(\bigvee\nolimits_i^{\uparrow} a_i\right)\right) = \nabla\emptyset \vee \nabla\{\bigvee\nolimits_i^{\uparrow} a_i\} = \bigvee\nolimits_i^{\uparrow}\psi(\Box a_i).$$

$\Box$ preserves top immediately from (∇2.0).

$\Box$ preserves binary meets:

$$\begin{aligned}\psi(\Box a_1) \wedge \psi(\Box a_2) &= \nabla\emptyset \vee (\nabla\{a_1\} \wedge \nabla\{a_2\})\\ &= \nabla\emptyset \vee \bigvee\{\nabla\beta \mid \beta \leq_C \{a_1\},\, \beta \leq_C \{a_2\}\}\\ &= \nabla\emptyset \vee \nabla\{a_1 \wedge a_2\} = \psi(\Box(a_1 \wedge a_2)).\end{aligned}$$

$\Diamond$ preserves joins:

$$\begin{aligned}\psi(\Diamond(\bigvee A)) &= \bigvee\{\nabla(\beta \cup \{\bigvee A\}) \mid \beta \in P_\omega L\}\\ &= \bigvee\{\nabla(\beta \cup \alpha) \mid \beta \in P_\omega L, \emptyset \neq \alpha \in P_\omega A\}\\ &= \bigvee\nolimits_{a\in A}\bigvee\{\nabla(\beta \cup \{a\}) \mid \beta \in P_\omega L\} = \bigvee\nolimits_{a\in A}\psi(\Diamond a).\end{aligned}$$

For the first mixed relation, and noting that $\nabla\emptyset \wedge \nabla(\beta \cup \{b\}) \leq \nabla\emptyset \wedge \nabla\{1\} = 0$, we have:

$$\begin{aligned}\psi(\Box a) \wedge \psi(\Diamond b) &= \bigvee\nolimits_{\beta\in P_\omega L}(\nabla\emptyset \vee \nabla\{a\}) \wedge \nabla(\beta \cup \{b\})\\ &= \bigvee\nolimits_{\beta\in P_\omega L}\nabla\{a\} \wedge \nabla(\beta \cup \{b\})\\ &= \bigvee\{\nabla\gamma \mid \exists\beta,\, \gamma \leq_C \{a\}, \gamma \leq_C \beta \cup \{b\}\}\\ &\leq \bigvee\nolimits_{\beta\in P_\omega L}\nabla(\beta \cup \{a \wedge b\}) = \psi(\Diamond(a \wedge b)).\end{aligned}$$

For the second:

$$\begin{aligned}\psi(\Box(a \vee b)) &= \nabla\emptyset \vee \nabla\{a \vee b\}\\ &= \nabla\emptyset \vee \nabla\{a\} \vee \nabla\{b\} \vee \nabla\{a, b\}\\ &\leq \psi(\Box a) \vee \psi(\Diamond b)\end{aligned}$$

since $\nabla\emptyset \vee \nabla\{a\} = \psi(\Box a)$ and $\nabla\{b\} \vee \nabla\{a, b\} \leq \psi(\Diamond b)$.

It remains to show that φ and ψ are mutually inverse.

$$\varphi(\psi(\Box a)) = \varphi(\nabla\emptyset \vee \nabla\{a\}) = \Box 0 \vee (\Box a \wedge \Diamond a) = \Box a$$

since $\Box 0 \wedge \Diamond a \leq \Diamond(0 \wedge a) = 0$.

Next, to show $\varphi(\psi(\Diamond a)) = \Diamond a$, we have

$$\begin{aligned}\varphi(\psi(\Diamond a)) &= \bigvee\nolimits_{\beta\in P_\omega L}\left(\Box(\bigvee\beta \vee a) \wedge \bigwedge\nolimits_{b\in\beta}\Diamond b \wedge \Diamond a\right)\\ &\leq \Diamond a\\ &= \Box(1 \vee a) \wedge \Diamond 1 \wedge \Diamond a = \varphi(\nabla\{1, a\}) \leq \varphi(\psi(\Diamond a)).\end{aligned}$$

Finally, to show $\psi(\varphi(\nabla\alpha)) = \nabla\alpha$, we have

$$\begin{aligned}\psi(\varphi(\nabla\alpha)) &= \psi\left(\Box(\bigvee\alpha) \wedge \bigwedge\nolimits_{a\in\alpha}\Diamond a\right)\\ &= (\nabla\emptyset \vee \nabla\{\bigvee\alpha\}) \wedge \bigwedge\nolimits_{a\in\alpha}\bigvee\nolimits_{\beta_a\in P_\omega L}\nabla(\beta \cup \{a\}).\end{aligned}$$

Now,

$$\begin{aligned}\textstyle\bigwedge_{a\in\alpha}\bigvee_{\beta_a\in P_\omega L}\nabla\left(\beta\cup\{a\}\right) &= \bigvee\left\{\nabla\gamma \mid \forall a\in\alpha,\ \exists\beta_a\in P_\omega L,\ \gamma\leq_C \beta_a\cup\{a\}\right\}\\ &= \bigvee\left\{\nabla\gamma \mid \gamma\leq_U \alpha\right\}.\end{aligned}$$

Also

$$\begin{aligned}\nabla\emptyset\wedge\bigvee\left\{\nabla\gamma\mid\gamma\leq_U\alpha\right\} &= \bigvee\left\{\nabla\delta\mid\delta\leq_C\emptyset,\delta\leq_U\alpha\right\}\\ &= \begin{cases}\nabla\alpha & \text{if } \alpha=\emptyset\\ 0 & \text{if } \alpha\neq\emptyset\end{cases}\\ \nabla\left\{\bigvee\alpha\right\}\wedge\bigvee\left\{\nabla\gamma\mid\gamma\leq_U\alpha\right\} &= \bigvee\left\{\nabla\delta\mid\delta\leq_C\left\{\bigvee\alpha\right\},\delta\leq_U\alpha\right\}\\ &= \nabla\left(\alpha\cup\left\{\bigvee\alpha\right\}\right)\\ &= \bigvee\left\{\nabla\left(\alpha\cup\alpha'\right)\mid\emptyset\neq\alpha'\in P_\omega\alpha\right\}\\ &= \begin{cases}0 & \text{if } \alpha=\emptyset\\ \nabla\alpha & \text{if } \alpha\neq\emptyset\end{cases}\end{aligned}$$

It follows that, whether α is empty or not, $\psi\left(\varphi\left(\nabla\alpha\right)\right)=\nabla\alpha$. □

3.4. **Categorical properties of the T-powerlocale.** In this section we discuss two categorical properties of the T-powerlocale construction. First we show how to extend the frame construction V_T to an endofunctor on the category Fr of frames. As a second topic we will see how the natural transformation $i\colon V_{P_\omega}\to V_{Id}$ (discussed in §2.5 as $i\colon V\to Id$) can be generalized to a natural transformation

$$\widehat{\rho}\colon V_T\to V_{T'},$$

for any natural transformation $\rho\colon T'\to T$ satisfying some mild conditions (where T and T' are two finitary, weak pullback preserving set functors).

3.4.1. *V_T is a functor.* We start with introducing a natural way to transform a frame homomorphism $f\colon\mathbb{L}\to\mathbb{M}$ into a frame homomorphism from $V_T\mathbb{L}$ to $V_T\mathbb{M}$. For that purpose we first prove the following technical lemma.

Lemma 3.16. *Let $T\colon\mathsf{Set}\to\mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor, let $\mathbb{L},\mathbb{M}$ be frames and let $f\colon\mathbb{L}\to\mathbb{M}$ be a frame homomorphism. Then the map $\nabla\circ Tf\colon TL\to V_TM$, i.e. $\alpha\mapsto\nabla(Tf)(\alpha)$, is compatible with the relations* ($\nabla1$), ($\nabla2$) *and* ($\nabla3$)*.*

Proof. We abbreviate $\heartsuit:=\nabla\circ Tf$, that is, for $\alpha\in TL$, we define $\heartsuit\alpha:=\nabla(Tf)(\alpha)$.

In order to prove that $\heartsuit$ is compatible with ($\nabla1$), we need to show that

$$\text{for all } \alpha,\beta\in TL\colon\ \alpha\,\overline{T}{\leq_{\mathbb{L}}}\,\beta \text{ implies } \heartsuit\alpha\leq_{V_T\mathbb{M}}\heartsuit\beta. \tag{9}$$

To see this, assume that $\alpha,\beta\in TL$ are such that $\alpha\,\overline{T}{\leq_{\mathbb{L}}}\,\beta$. From this it follows by Lemma 2.7 and the assumption that f is a frame homomorphism, that $(Tf)(\alpha)\,\overline{T}{\leq_{\mathbb{M}}}\,(Tf)(\beta)$. Then by $(\nabla1)_{\mathbb{M}}$ we obtain that $\heartsuit\alpha\leq_{V_T\mathbb{M}}\heartsuit\beta$, as required.

Proving compatibility with ($\nabla2$) boils down to showing

$$\text{for all } \Gamma\in P_\omega TL\colon\ \bigwedge_{\alpha\in\Gamma}\heartsuit\alpha\leq\bigvee\{\heartsuit(T\textstyle\bigwedge)(\Psi)\mid\Psi\in SRD(\Gamma)\}. \tag{10}$$

For this purpose, given $\Gamma \in P_\omega TL$, let $\Gamma' \in P_\omega TM$ denote the set $\Gamma' := (P_\omega Tf)(\Gamma) = \{(Tf)(\alpha) \mid \alpha \in \Gamma\}$. Then we may observe

$$\begin{aligned}
\bigwedge_{\alpha\in\Gamma} \heartsuit\alpha &= \bigvee\{\nabla(T\textstyle\bigwedge)(\Psi) \mid \Psi \in SRD(\Gamma')\} && (\nabla 1)\\
&\leq \bigvee\{\nabla(T\textstyle\bigwedge)(TP_\omega f)(\Phi) \mid \Phi \in SRD(\Gamma)\} && (\text{Lemma 3.9})\\
&= \bigvee\{\nabla(Tf)(T\textstyle\bigwedge)(\Phi) \mid \Phi \in SRD(\Gamma)\} && (\dagger)\\
&= \bigvee\{\heartsuit(T\textstyle\bigwedge)(\Phi) \mid \Phi \in SRD(\Gamma)\} && (\text{definition of } \heartsuit)
\end{aligned}$$

Here the identity marked (†) is easily justified by f being a homomorphism: it follows from $f \circ \bigwedge = \bigwedge \circ (P_\omega f)$ and functoriality of T that $(Tf) \circ (T\bigwedge) = (T\bigwedge) \circ (TP_\omega f)$.

Finally, for compatibility with $(\nabla 3)$ we need to verify that

$$\text{for all } \Phi \in TPL\colon \heartsuit(T\textstyle\bigvee)(\Phi) \leq \bigvee\{\heartsuit\beta \mid \beta \mathrel{\overline{T}{\in}} \Phi\}. \tag{11}$$

To prove this, we calculate for a given $\Phi \in TPL$:

$$\begin{aligned}
\heartsuit(T\textstyle\bigvee)(\Phi) &= \nabla(Tf)(T\textstyle\bigvee)(\Phi) && (\text{definition of } \heartsuit)\\
&= \nabla(T\textstyle\bigvee)(TPf)(\Phi) && (f \text{ a frame homomorphism})\\
&\leq \bigvee\{\nabla\beta \mid \beta \mathrel{\overline{T}{\in}} (TPf)(\Phi)\} && (\nabla 3)_{\mathbb{M}}\\
&= \bigvee\{\nabla(Tf)(\gamma) \mid \gamma \mathrel{\overline{T}{\in}} \Phi\} && (\ddagger)\\
&= \bigvee\{\heartsuit\gamma \mid \gamma \mathrel{\overline{T}{\in}} \Phi\} && (\text{definition of } \heartsuit)
\end{aligned}$$

Here the identity (‡) follows from the observation that for all $\beta \in TM$ and $\Phi \in TPL$, we have $\beta \mathrel{\overline{T}{\in}} (TPf)(\Phi)$ iff β is of the form $\beta = (Tf)(\gamma)$ for some $\gamma \in TL$. Using Fact 2.6, this is easily derived from the observation that for $b \in M$ and $A \in PL$, we have $b \in (Pf)A$ iff $b = f(c)$ for some $c \in A$. □

Lemma 3.16 justifies the following definition.

Definition 3.17. Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor and let $f\colon \mathbb{L} \to \mathbb{M}$ be a frame homomorphism. We define $V_T f\colon V_T\mathbb{L} \to V_T\mathbb{M}$ to be the unique frame homomorphism extending

$$\nabla \circ Tf\colon TL \to V_T M.$$

Theorem 3.18. *Let T be a standard, finitary, weak pullback-preserving functor. Then the operation V_T defined above is an endofunctor on the category* Fr.

Proof. Since for an arbitrary $f\colon \mathbb{L} \to \mathbb{M}$ we have ensured by definition that $V_T F$ is a frame homomorphism from $V_T\mathbb{L}$ to $V_T\mathbb{M}$, it is left to show that V_T maps the identity map of a frame to the identity map of its T-powerlocale, and distributes over function composition. We confine our attention to the second property.

Let $f\colon \mathbb{K} \to \mathbb{L}$ and $g\colon \mathbb{L} \to \mathbb{M}$ be two frame homomorphisms. In order to show that $V_T(g \circ f) = V_T g \circ V_T f$, first recall that $V_T(g \circ f)$ is by definition the unique frame homomorphism extending the map $\nabla_{\mathbb{M}} \circ T(g \circ f)\colon TK \to V_T\mathbb{M}$. Hence, it suffices to prove that the map $V_T g \circ V_T f$, which is obviously a frame homomorphism, extends $\nabla_{\mathbb{M}} \circ T(g \circ f)$. But it is easy to see why this is the case: given an arbitrary element $\alpha \in TK$, a straightforward unraveling of definitions shows that

$$(V_T g \circ V_T f)(\alpha) = V_T g(\nabla_{\mathbb{L}}(Tf)(\alpha)) = \nabla_{\mathbb{M}}(Tg)(Tf)(\alpha) = \nabla_{\mathbb{M}} T(g \circ f)(\alpha),$$

as required. □

3.4.2. *Natural transformations between V_T and $V_{T'}$.* Now that we have seen how each finitary, weak pullback preserving set functor T induces a functor V_T on the category of frames, we investigate the relation *between* two such functors $V_T, V_{T'}$. In fact, we have already seen an example of this: recall that in §2.5 we mentioned Johnstone's result (Johnstone, 1985) that the standard Vietoris functor V is in fact a *comonad* on the category of frames. In our nabla-based presentation of this functor as $V = V_{P_\omega}$, thinking of the identity functor on the category Fr as the Vietoris functor V_{Id}, we can see the *counit* of this comonad as a natural transformation

$$i\colon V_{P_\omega} \to V_{Id},$$

given by $i_{\mathbb{L}}\colon \nabla A \mapsto \bigwedge A$. More precisely, we can show that the map $\heartsuit\colon P_\omega L \to L$ given by $\heartsuit A := \bigwedge A$ is compatible with the ∇-axioms, and hence can be uniquely extended to the homomorphism $i_{\mathbb{L}}$; subsequently we can show that this i is natural in $\mathbb{L}$. Recall that in the case of a concrete topological space (X,τ), this counit corresponds on the dual side to the singleton map $\sigma_X : s \mapsto \{s\}$ which provides an embedding of a compact Hausdorff topology into its Vietoris space.

We will now see how to generalize this picture, of the natural transformation $i\colon V_{P_\omega} \to V_{Id}$ being induced by the singleton natural transformation $\sigma\colon Id \to P_\omega$, to a more general setting. First consider the following definition.

Definition 3.19. Let T and T' be standard, finitary, weak pullback-preserving functors. A natural transformation $\rho\colon T' \to T$ is said to *respect relation lifting* if for any relation $R \subseteq X \times Y$ we have, for all $\alpha' \in T'X$ and $\beta' \in T'Y$:

$$\text{if } \alpha' \,\overline{T'}R\, \beta' \text{ then } \rho_X(\alpha') \,\overline{T}R\, \rho_Y(\beta'). \tag{12}$$

We call ρ *base-invariant* if it commutes with *Base*, that is,

$$Base^{T'} = Base^{T} \circ \rho. \tag{13}$$

for any set X.

Example 3.20. We record three examples of base-invariant natural transformations which respect relation lifting.

(1) The base transformation $Base^T : T \to P_\omega$;
(2) The singleton natural transformation $\sigma\colon \mathrm{Id} \to P_\omega$, which is in fact a special case of (1);
(3) The diagonal map δ (given by $\delta_X : x \mapsto (x,x)$); it is straightforward to check that as a natural transformation, $\delta\colon Id \to Id \times Id$ also satisfies both properties of Definition 3.19.

As we will see now, every base-invariant natural transformation $\rho\colon T' \to T$ that respects relation lifting, induces a natural transformation $\widehat{\rho}\colon V_T \to V_{T'}$. In particular, the natural transformation $i\colon V \to Id$ can be seen as $i = \widehat{\sigma}$, where $\sigma\colon Id \to P_\omega$ is the singleton transformation discussed above.

Theorem 3.21. *Let T and T' be standard, finitary, weak pullback-preserving functors, assume that $\rho\colon T' \to T$ is a base-invariant natural transformation that respects relation lifting, and let $\mathbb{L}$ be a frame. Then the map from TL to $V_{T'}L$ given by*

$$\alpha \mapsto \bigvee\{\nabla\alpha' \mid \alpha' \in T'L, \rho(\alpha') \,\overline{T}{\leq}\, \alpha\}$$

specifies a frame homomorphism

$$\widehat{\rho}_{\mathbb{L}}\colon V_T\mathbb{L} \to V_{T'}\mathbb{L}$$

which is natural in $\mathbb{L}$.

Proof. We let $\heartsuit\colon TL \to L$ denote the map given in the statement of the Theorem, that is, $\heartsuit\alpha := \bigvee\{\nabla\alpha' \mid \alpha' \in T'L, \rho(\alpha') \,\overline{T}{\leq}\, \alpha\}$. We will first prove that this map is compatible with, respectively, (∇1), (∇2) and (∇3), and then turn to the naturality of the induced frame homomorphism.

Claim 1. The map $\heartsuit$ is compatible with (∇1).

Proof of Claim To show that $\heartsuit$ is compatible with (∇1), take two elements $\alpha, \beta \in TL$ such that $\alpha\,\overline{T}{\leq}\,\beta$. Then for any $\alpha' \in T'L$ such that $\rho(\alpha')\,\overline{T}{\leq}\,\alpha$, by transitivity of $\overline{T}{\leq}$ (Fact 2.6(5)), we obtain that $\rho(\alpha')\,\overline{T}{\leq}\,\beta$. From this it is immediate that $\heartsuit\alpha \leq \heartsuit\beta$, as required.

Claim 2. The map $\heartsuit$ is compatible with (∇2).

Proof of Claim For compatibility with (∇2), it suffices to show compatibility with ($\nabla 2'$). That is, for $\Gamma \in P_\omega TL$, we will verify that

$$\bigwedge\{\heartsuit\gamma \mid \gamma \in \Gamma\} \leq \bigvee\{\heartsuit\beta \mid \beta\,\overline{T}{\leq}\,\gamma, \text{ for all } \gamma \in \Gamma\}. \tag{14}$$

We start with rewriting the left hand side of (14) into

$$\begin{aligned}
\bigwedge\{\heartsuit\gamma \mid \gamma \in \Gamma\} &= \bigwedge\left\{\bigvee\left\{\nabla\gamma' \mid \rho(\gamma')\,\overline{T}{\leq}\,\gamma\right\} \mid \gamma \in \Gamma\right\} && \text{(definition of } \heartsuit\text{)}\\
&= \bigvee\left\{\bigwedge\left\{\varphi_\gamma \mid \gamma \in \Gamma\right\} \mid \varphi \in \mathcal{C}_\Gamma\right\} && \text{(frame distributivity)}
\end{aligned}$$

where we define $\mathcal{C}_\Gamma := \{\varphi\colon \Gamma \to T'L \mid \rho(\varphi_\gamma)\,\overline{T}{\leq}\,\gamma, \text{ for all } \gamma \in \Gamma\}$.

For any map $\varphi \in \mathcal{C}_\Gamma$ we may calculate

$$\begin{aligned}
&\bigwedge\left\{\varphi_\gamma \mid \gamma \in \Gamma\right\}\\
&= \bigvee\{\nabla\gamma' \mid \gamma'\,\overline{T'}{\leq}\,\varphi_\gamma, \forall\gamma \in \Gamma\} && (\nabla 2')\\
&\leq \bigvee\{\nabla\gamma' \mid \rho(\gamma')\,\overline{T}{\leq}\,\rho(\varphi_\gamma), \forall\gamma \in \Gamma\} && (\rho \text{ respects relation lifting})\\
&\leq \bigvee\{\nabla\gamma' \mid \rho(\gamma')\,\overline{T}{\leq}\,\gamma, \forall\gamma \in \Gamma\} && (\varphi \in \mathcal{C}_\Gamma, \text{ transitivity of } \overline{T}{\leq})\\
&= \bigvee\left\{\bigvee\{\nabla\gamma' \mid \rho(\gamma')\,\overline{T}{\leq}\,\beta\} \mid \beta\,\overline{T}{\leq}\,\gamma, \forall\gamma \in \Gamma\right\} && (\text{associativity of } \textstyle\bigvee)\\
&= \bigvee\{\heartsuit\beta \mid \beta\,\overline{T}{\leq}\,\gamma, \forall\gamma \in \Gamma\} && (\text{definition of } \heartsuit)
\end{aligned}$$

From the above calculations, (14) is immediate.

Claim 3. The map $\heartsuit$ is compatible with (∇3).

Proof of Claim We need to show, for an arbitrary but fixed set $\Phi \in TPL$, that

$$\heartsuit(T\textstyle\bigvee)(\Phi) = \bigvee\{\heartsuit\alpha \mid \alpha\,\overline{T}{\in}\,\Phi\}. \tag{15}$$

By definition, on the left hand side of (15) we find

$$\heartsuit(T\textstyle\bigvee)(\Phi) = \bigvee\{\nabla\beta' \mid \rho(\beta')\,\overline{T}{\leq}\,(T\bigvee)(\Phi)\},$$

while on the right hand side we obtain, by definition of $\heartsuit$,

$$\begin{aligned}
\bigvee\{\heartsuit\alpha \mid \alpha\,\overline{T}{\in}\,\Phi\} &= \bigvee\left\{\bigvee\{\nabla\alpha' \mid \rho(\alpha')\,\overline{T}{\leq}\,\alpha\} \mid \alpha\,\overline{T}{\in}\,\Phi\right\}\\
&= \bigvee\left\{\nabla\alpha' \mid \rho(\alpha')\,\overline{T}(\leq;\in)\,\Phi\right\}
\end{aligned}$$

where the latter equality is by associativity of $\bigvee$, and the compositionality of relation lifting (Fact 2.6(5)).

As a consequence, in order to establish the compatibility of $\heartsuit$ with (∇3), it suffices to show that

$$\nabla\beta' \leq \bigvee\left\{\nabla\alpha' \mid \rho(\alpha')\,\overline{T}(\leq;\in)\,\Phi\right\}, \text{ for any } \beta' \text{ with } \rho(\beta')\,\overline{T}{\leq}\,(T\textstyle\bigvee)(\Phi). \tag{16}$$

Let β' be an arbitrary element of TL such that $\rho(\beta')\,\overline{T}{\leq}\,(T\bigvee)(\Phi)$. Our goal will be to find a set $\Phi' \in T'PL$ satisfying (20), (21) and (22) below: clearly this will satisfy to prove (16).

By Fact 2.8 we obtain that

$$\mathit{Base}^T(\rho\beta')\,\overline{P}{\leq}\,\mathit{Base}^T((T\textstyle\bigvee)(\Phi)) = (P\bigvee)\,\mathit{Base}^T(\Phi),$$

and since ρ is base-invariant, we have $Base^{T'}(\beta') = Base^{T}(\rho\beta')$. Combining these facts we see that $Base^{T'}(\beta')\ \overline{P}{\leq}\ (P\bigvee)\ Base^{T}(\Phi)$. This motivates the definition of the following map $\mathcal{H}\colon\ Base^{T'}(\beta') \to P_\omega PL$:

$$\mathcal{H}(b) := \{B \in Base^{T}(\Phi) \mid b \leq \bigvee B\}.$$

From the definitions it is immediate that

$$\text{for all } b \in Base^{T'}(\beta') :\ b \leq \bigwedge \left\{\bigvee B \mid B \in \mathcal{H}(b)\right\}. \tag{17}$$

Also, given a set $\mathcal{B} \in P_\omega PL$, let $\mathcal{C}_B$ be the collection of choice functions on $\mathcal{B}$, that is:

$$\mathcal{C}_\mathcal{B} := \{f\colon \mathcal{B} \to L \mid f(B) \in B \text{ for all } B \in \mathcal{B}\}.$$

Then it follows by frame distributivity that

$$\bigwedge \left\{\bigvee B \mid B \in \mathcal{B}\right\} = \bigvee \left\{\bigwedge (Pf)(\mathcal{B}) \mid f \in \mathcal{C}_\mathcal{B}\right\}. \tag{18}$$

Define the map $K\colon P_\omega PL \to PL$ by

$$K(\mathcal{B}) := \left\{\bigwedge (Pf)(\mathcal{B}) \mid f \in \mathcal{C}_\mathcal{B}\right\},$$

then it follows from (17), (18) and the definitions that

$$\text{for all } b \in Base^{T'}(\beta') :\ b \leq \bigvee K(\mathcal{H}(b)). \tag{19}$$

As a corollary, if we define

$$\Phi' := (T'K)(T'\mathcal{H})(\beta'),$$

then it follows from (19), by the properties of relation lifting, that $\beta'\ \overline{T'}{\leq}\ (T'\bigvee)(\Phi')$, so that an application of (∇1) yields

$$\nabla\beta' \leq \nabla(T'\bigvee)(\Phi'). \tag{20}$$

Also, on the basis of an application of (∇3) we may conclude that

$$\nabla(T'\bigvee)(\Phi') \leq \bigvee\{\nabla\gamma' \mid \gamma'\ \overline{T'}{\in}\ \Phi'\}. \tag{21}$$

This means that we are done with the proof of (16) if we can show that

$$\text{for any } \gamma' \in T'L,\ \text{ if } \gamma'\ \overline{T'}{\in}\ \Phi' \text{ then } \rho(\gamma')\ \overline{T}(\leq ; \in)\ \Phi. \tag{22}$$

For a proof of (22), let γ' be an arbitrary T'-lifted member of Φ' and recall that $\Phi' = (TK)(T\mathcal{H})(\beta')$. Then it follows by the assumption that ρ respects relation lifting, that $\rho(\gamma')\ \overline{T}{\in}\ \rho(\Phi') = (TK)(T\mathcal{H})(\rho(\beta'))$. Given our assumption on β', this means that the relation between $\rho(\gamma')$ and Φ can be summarized as

$$\rho(\gamma')\ \overline{T}{\in}\ (TK)(T\mathcal{H})(\beta) \text{ and } \beta\ \overline{T}{\leq}\ (T\bigvee)(\Phi) \text{ for some } \beta \in T\,Base^{T'}(\beta'), \tag{23}$$

where for β we may take $\rho(\beta')$.

Returning to the ground level, observe that for any $c \in L$, $A \in Base^{T}(\Phi)$, we have

$$\text{if } c \in K\mathcal{H}(b) \text{ and } b \leq \bigvee A,\ \text{ for some } b \in Base^{T'}(\beta'),\ \text{ then } c\ (\leq ; \in)\ A. \tag{24}$$

To see why this is the case, assume that $c \in K\mathcal{H}(b)$ and $b \leq \bigvee A$, for some $b \in Base^{T'}(\beta')$. Then by definition of $\mathcal{H}$ we find $A \in \mathcal{H}(b)$, while $c \in K\mathcal{H}(b)$ simply means that $c = \bigwedge\{f(B) \mid B \in \mathcal{H}(b)\}$, for some $f \in \mathcal{C}_{\mathcal{H}(b)}$. But then it is immediate that $c \leq f(A)$, while $f(A) \in A$ by definition of $\mathcal{C}_{\mathcal{H}(b)}$. Thus $f(A)$ is the required element witnessing that $c\ (\leq ; \in)\ A$.

But by the properties of relation lifting, we may derive from (24) that

$$\begin{aligned} &\text{if } \gamma\ \overline{T}{\in}\ (TK)(T\mathcal{H})(\beta) \text{ and } \beta\ \overline{T}{\leq}\ (T\bigvee)(\Phi) \text{ for some } \beta \in T\,Base^{T'}(\beta'),\\ &\text{then } \gamma\ \overline{T}(\leq ; \in)\ \Phi, \end{aligned} \tag{25}$$

so that it is immediate by (23) that $\rho(\gamma')\ \overline{T}(\leq ; \in)\ \Phi$. This proves (22).

As mentioned already, the compatibility of $\heartsuit$ with (∇3) is immediate by (20), (21) and (22), and so this finishes the proof of Claim 3.

As a corollary of the Claims 1–3, we may uniquely extend $\heartsuit$ to a homomorphism $\widehat{\rho}_{\mathbb{L}}\colon V_T\mathbb{L} \to V_{T'}\mathbb{L}$. Clearly then, in order to prove the theorem it suffices to prove the following claim.

Claim 4. The family of homomorphisms $\widehat{\rho}_{\mathbb{L}}$ constitutes a natural transformation $\widehat{\rho}\colon V_T \to V_{T'}$.

Proof of Claim Given two frames $\mathbb{L}$ and $\mathbb{M}$ and a frame homomorphism $f\colon \mathbb{L} \to \mathbb{M}$, we need to show that the following diagram commutes:

$$\begin{tikzcd} V_T\mathbb{L} \arrow[r, "\widehat{\rho}_{\mathbb{L}}"] \arrow[d, "V_T f"'] & V_{T'}\mathbb{L} \arrow[d, "V_{T'} f"] \\ V_T\mathbb{M} \arrow[r, "\widehat{\rho}_{\mathbb{M}}"] & V_{T'}\mathbb{M} \end{tikzcd}$$

To show this, take an arbitrary element $\alpha \in TL$, and consider the following calculation:

$$\begin{array}{lll}
& (V_{T'}f)(\widehat{\rho}_{\mathbb{L}}(\nabla\alpha)) & \\
= & (V_{T'}f)(\heartsuit\alpha) & \text{(definition of } \widehat{\rho}_{\mathbb{L}})) \\
= & (V_{T'}f)\left(\bigvee\left\{\nabla\beta' \mid \rho_L(\beta')\,\overline{T}{\leq}\,\alpha\right\}\right) & \text{(definition of } \heartsuit)) \\
= & \bigvee\left\{(V_T'f)(\nabla\beta') \mid \rho_L(\beta')\,\overline{T}{\leq}\,\alpha\right\} & (V_{T'}f \text{ is a frame homomorphism}) \\
= & \bigvee\left\{\nabla(T'f)(\beta') \mid \rho_L(\beta')\,\overline{T}{\leq}\,\alpha\right\} & \text{(definition of } V_{T'}f)) \\
= & \bigvee\left\{\nabla\delta' \mid \rho_M(\delta')\,\overline{T}{\leq}\,(Tf)(\alpha)\right\} & (\dagger) \\
= & \heartsuit(Tf)(\alpha) & \text{(definition of } \heartsuit)) \\
= & \widehat{\rho}_{\mathbb{M}}(\nabla(Tf)(\alpha)) & \text{(definition of } \widehat{\rho}_{\mathbb{M}})) \\
= & \widehat{\rho}_{\mathbb{M}}((V_Tf)(\nabla\alpha)) & \text{(definition of } V_Tf))
\end{array}$$

Here the crucial step, marked (†), is proved by establishing the two respective inequalities, as follows. For the inequality $\leq$, it is straightforward to show that the set of joinands on the left hand side is included in that on the right hand side, and this follows from

$$\rho_L(\beta')\,\overline{T}{\leq}\,\alpha \text{ implies } \rho_M((T'f)(\beta'))\,\overline{T}{\leq}\,(Tf)(\alpha). \tag{26}$$

To prove (26), suppose that $\rho_L(\beta')\,\overline{T}{\leq}\,\alpha$; then it follows by the fact that f is a homomorphism, and hence, monotone, that $(Tf)(\rho_L(\beta'))\,\overline{T}{\leq}\,(Tf)(\alpha)$. But since ρ is a natural transformation, we also have $(Tf)(\rho_L(\beta')) = \rho_M(T'f)(\beta')$, and from this (26) is immediate.

In order to prove the opposite inequality

$$\bigvee\left\{\nabla\delta' \mid \rho_M(\delta')\,\overline{T}{\leq}\,(Tf)(\alpha)\right\} \leq \bigvee\left\{\nabla(T'f)(\beta') \mid \rho_L(\beta')\,\overline{T}{\leq}\,\alpha\right\}, \tag{27}$$

fix an arbitrary element $\delta' \in TL$ such that $\rho_M(\delta')\,\overline{T}{\leq}\,(Tf)(\alpha)$.

Define the map $h\colon\ \mathit{Base}^{T'}(\delta') \to L$ by putting

$$h(d) := \bigwedge\{a \in \mathit{Base}^{T}(\alpha) \mid d \leq f(a)\}.$$

Then for all $d \in \mathit{Base}^{T'}(\delta')$ and all $a \in \mathit{Base}^{T}(\alpha)$, we find that $d \leq fa$ implies $hd \leq a$; this can be expressed by the relational inclusion

$$\mathit{Gr}\,f\,;\geq;\mathit{Gr}\,h\ \subseteq\ \geq$$

so that by the properties of relation lifting we may conclude that $Gr\,(Tf)\,;\overline{T}\!\geq\,;Gr\,(Th)\ \subseteq\ \overline{T}\!\geq$, which is just another way of saying that, for all $\delta \in T\,Base^{T'}(\delta')$, we have

$$\delta\ \overline{T}\!\leq (Tf)(\alpha) \text{ only if } (Th)(\delta)\ \overline{T}\!\leq \alpha. \tag{28}$$

Now define

$$\beta' := (T'h)(\delta'),$$

then we may conclude from the fact that ρ respects relation lifting that $\rho_L(\beta') = (Th)\rho_M(\delta')$, and so by the assumption that $\rho_M(\delta')\ \overline{T}\!\leq (Tf)(\alpha)$, we obtain by (28) that

$$\rho_L(\beta')\ \overline{T}\!\leq \alpha. \tag{29}$$

Similarly, from the fact that $d \leq fhd$, for each $d \in Base^{T'}(\delta')$, we may derive that $\delta'\ \overline{T'}\!\leq (T'f)(\beta')$, and so by ($\nabla$1) we may conclude that

$$\nabla\delta' \leq \nabla(T'f)(\beta'). \tag{30}$$

Finally, (26) is immediate by (25) and (30).

This finishes the proof of Claim 4.

□

Remark 3.22. The definition of the $\widehat{\rho}_{\mathbb{L}}\colon V_T\mathbb{L} \to V_{T'}\mathbb{L}$, using the assignment

$$\alpha \mapsto \bigvee\{\nabla\alpha' \mid \alpha' \in T'L, \rho(\alpha')\ \overline{T}\!\leq \alpha\},$$

is very similar to that of a right adjoint. If it were the case that $\widehat{\rho}_{\mathbb{L}}$ preserved *all* meets, then the adjoint functor theorem would allow us to define its left adjoint. However, we only have a proof that $\widehat{\rho}_{\mathbb{L}}\colon V_T\mathbb{L} \to V_{T'}\mathbb{L}$ preserves *finite* conjunctions, so it is not at all obvious at this point if there even is a left adjoint to $\widehat{\rho}_{\mathbb{L}}$. This is an interesting question for future work.

3.5. **T-powerlocales via flat sites.** In this subsection, we will show that $V_T\mathbb{L}$, the T-powerlocale of a given frame $\mathbb{L}$, has a flat site presentation as $V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$. It then follows by the Flat site Coverage Theorem that every element of $V_T\mathbb{L}$ has a disjunctive normal form, and that the suplattice reduct of $V_T\mathbb{L}$ has a presentation defined only in terms of the order $\overline{T}\!\leq$ and the lifted join function $T\bigvee\colon TPL \to TL$.

Recall that $\langle X, \sqsubseteq, \lhd_0\rangle$ is a flat site if $\langle X, \sqsubseteq\rangle$ is a pre-order and $\lhd_0$ is a basic cover relation compatible with $\sqsubseteq$. In that case, we know that $\langle X, \sqsubseteq, \lhd_0\rangle$ presents a frame $\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0\rangle$, and that if we denote the insertion of generators by $\heartsuit\colon X \to \mathrm{Fr}\langle X, \sqsubseteq, \lhd_0\rangle$, then

$$\begin{aligned}\mathrm{Fr}\langle X, \sqsubseteq, \lhd_0\rangle \simeq \mathrm{Fr}\langle X \mid\ \ & \heartsuit a \leq \heartsuit b \quad (a \sqsubseteq b),\\ & 1 = \textstyle\bigvee\{\heartsuit a \mid a \in X\}\\ & \heartsuit a \wedge \heartsuit b = \textstyle\bigvee\{\heartsuit c \mid c \sqsubseteq a, c \sqsubseteq b\}\\ & \heartsuit a \leq \textstyle\bigvee\{\heartsuit b \mid b \in A\} \quad (a \lhd_0 A)\rangle.\end{aligned}$$

Observe that this is very similar to our presentation of $V_T\mathbb{L}$ from Corollary 3.6 using (∇1), ($\nabla 2'$) and (∇3), namely

$$\begin{aligned}V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL \mid\ \ & \nabla\alpha \leq \nabla\beta \quad (\alpha\ \overline{T}\!\leq \beta),\\ & \textstyle\bigwedge_\Gamma \nabla\gamma = \bigvee\{\nabla\delta \mid \forall\gamma \in \Gamma,\ \delta\ \overline{T}\!\leq \gamma\} \quad (\Gamma \in TP_\omega L)\\ & \nabla T\textstyle\bigvee(\Phi) \leq \bigvee\{\nabla\beta \mid \beta \in \lambda^T(\Phi)\} \quad (\Phi \in TPL)\rangle.\end{aligned}$$

We will see below that if we define a cover relation $\lhd_0^{\mathbb{L}}$ which is inspired by (∇3), then we obtain a flat site $\langle TL, \overline{T}\!\leq, \lhd_0^{\mathbb{L}}\rangle$, and this flat site presents $V_T\mathbb{L}$.

So how do we go about defining a basic cover relation $\lhd_0^{\mathbb{L}} \subseteq TL \times PTL$ so we can give a presentation of $V_T\mathbb{L}$? Intuitively, we would like to take the T-lifting of the relation $\{(a, A) \in L \times PL \mid a \leq \bigvee A\} = \leq\,;(Gr\bigvee)^{\smile}$. However, the T-lifting of this relation is of type $TL \times TPL$, while a basic cover relation on $\langle TL, \overline{T}\!\leq\rangle$ should be of type $TL \times PTL$. We solve this by involving the natural transformation

$\lambda^T\colon TP \to PT$, given by $\lambda^T(\Phi) := \{\beta \in TL \mid \beta \,\overline{T}{\in}\, \Phi\}$, assigning to each $\Phi \in TPL$ the set of its lifted members. That is, we define

$$\lhd_0^{\mathbb{L}} \ := \ \{(\alpha, \lambda^T(\Phi)) \in L \times PTL \mid \alpha \,\overline{T}{\leq}\, T\bigvee(\Phi)\}.$$

In other words: we put $\alpha \lhd_0^{\mathbb{L}} \Gamma$ iff Γ is of the form $\lambda^T(\Phi)$ for some $\Phi \in TPL$ such that $\alpha \,\overline{T}{\leq}\, (T\bigvee)\Phi$. Two tasks lie ahead of us: first, we must show that $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is a flat site, meaning that $\lhd_0$ is compatible with $\overline{T}{\leq}$. Second, we must show that $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ presents $V_T\mathbb{L}$. The following technical observation about the relation $\alpha \,\overline{T}{\leq}\, T\bigvee(\Phi)$ is the main reason why $V_T\mathbb{L}$ admits a flat site presentation. The reason for introducing a $\wedge$-semilattice $\mathbb{M}$ below will become apparent in §4.3.

Lemma 3.23. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor, let $\mathbb{L}$ be a frame and let $\mathbb{M}$ be a $\wedge$-subsemilattice of $\mathbb{L}$. Then for all $\alpha \in TM$ and $\Phi \in TPM$ such that $\alpha \,\overline{T}{\leq}\, T\bigvee(\Phi)$, there exists $\Phi' \in TPM$ such that*

1. *$\alpha \,\overline{T}{\leq}\, T\bigvee(\Phi')$;*
2. *$\Phi' \,\overline{T}{\subseteq}\, T{\downarrow_L} \circ T\eta(\alpha)$;*
3. *$\Phi' \,\overline{T}{\subseteq}\, T{\downarrow_L}(\Phi)$*

Proof. First, we define the following relation on $M \times PM$:

$$R := \{(a, A) \in M \times PM \mid a \leq \bigvee A\} = (\leq ; (Gr \bigvee)^{\smile})\restriction_{M \times PM}.$$

Consider the span $M \xleftarrow{p_1} R \xrightarrow{p_2} PM$. We define the following function $f\colon R \to R$:

$$f\colon (a, A) \mapsto (a, a \wedge A),$$

where $a \wedge A := \{a \wedge b \mid b \in A\}$. To see why this function is well-defined, first observe that $a \wedge A \in PM$ because $\mathbb{M}$ is a $\wedge$-subsemilattice of $\mathbb{L}$. Moreover, by frame distributivity, we see that if $(a, A) \in R$, i.e. if $a \leq \bigvee A$, then also $a \leq \bigvee(a \wedge A)$, so that $(a, a \wedge A) \in R$. Now observe that $f\colon R \to R$ satisfies an equation and two inequations: for all $(a, A) \in R$,

$$\begin{aligned}
&p_1 \circ f(a, A) = a = p_1(a, A), && \text{by def. of } f,\\
&p_2 \circ f(a, A) = a \wedge A \subseteq_L {\downarrow_L}\{a\} = {\downarrow_L} \circ \eta_L \circ p_1(a, A), && \text{since } \forall b \in A,\ a \wedge b \leq a,\\
&p_2 \circ f(a, A) = a \wedge A \subseteq_L {\downarrow_L} A = {\downarrow_L} \circ p_2(a, A) && \text{since } \forall b \in A,\ a \wedge b \leq b \in A.
\end{aligned}$$

Now consider the lifted diagram

$$TM \xleftarrow{Tp_1} TR \xrightarrow{Tp2} TPM.$$

It follows from Lemma 2.7 and the equation/inequations above that for each $\delta \in TR$, we have

$$Tp_1 \circ Tf(\delta) = Tp_1(\delta), \tag{31}$$

$$Tp_2 \circ Tf(\delta) \,\overline{T}{\subseteq_L}\, T{\downarrow_L} \circ T\eta_L \circ Tp_1(\delta), \tag{32}$$

$$Tp_2 \circ Tf(\delta) \,\overline{T}{\subseteq_L}\, T{\downarrow_L} \circ Tp_2(\delta) \tag{33}$$

Now recall that by Fact 2.6,

$$\overline{T}{\leq} ; Gr(T\bigvee)^{\smile} = \overline{T}(\leq ; (Gr \bigvee)^{\smile}) = \overline{T}R,$$

so we see that $\alpha \,\overline{T} \leq T\bigvee(\Phi)$ iff $\alpha \,\overline{T}R\, \Phi$. So let $\alpha \in TM$ and $\Phi \in TPM$ such that $\alpha \,\overline{T}{\leq}\, T\bigvee(\Phi)$, i.e. such that $\alpha \,\overline{T}R\, \Phi$; we will show that there is a $\Phi' \in TPM$ satisfying properties (1)–(3). First, observe that by definition of relation lifting, there must exist some $\delta \in TR$ such that

$$Tp_1(\delta) = \alpha \text{ and } Tp_2(\delta) = \Phi.$$

We claim that $\Phi' := Tp_2 \circ Tf(\delta)$ satisfies properties (1)–(3). We know by definition of relation lifting that $(Tp_1 \circ Tf(\delta))\ \overline{T}R\ (Tp_2 \circ Tf(\delta))$. Since

$$\begin{aligned} Tp_1 \circ Tf(\delta) &= Tp_1(\delta) && \text{by (31),} \\ &= \alpha && \text{by assumption,} \end{aligned}$$

it follows that $\alpha\ \overline{T}R\ \Phi'$, i.e. $\alpha\ \overline{T}{\leq}\ T\bigvee(\Phi')$; we conclude that (1) holds. Moreover, it follows immediately from (32) that (2) holds. Similarly, it follows immediately from (33) that (3) holds. □

In the lemma above, we use the lifted inclusion relation $T{\subseteq}$ and the lifted downset function $T{\downarrow}$. In the lemma below we record some elementary observations about the interaction between $T{\subseteq}$, $T{\downarrow}$ and the natural transformation $\lambda^T : TP \to PT$.

Lemma 3.24. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor, let $\langle X, \sqsubseteq\rangle$ be a pre-order, let $\alpha \in TX$ and let $\Phi, \Phi' \in TPX$. Then*

(1) *${\downarrow}_{TX}\lambda^T(\Phi) = \lambda^T\,(T{\downarrow}_X(\Phi))$;*
(2) *${\downarrow}_{TX}\{\alpha\} = \lambda^T\,(T{\downarrow}_X \circ T\eta_X(\alpha))$;*
(3) *If $\Phi'\ \overline{T} \subseteq_X \Phi$, then also $\lambda^T(\Phi') \subseteq \lambda^T(\Phi)$.*

Proof. (1). For all $a \in X$ and all $A \in PX$, we have $a \leq ; \in A$ iff $a \in {\downarrow}_X A$. Consequently,

$$\forall \alpha \in TL, \forall \Phi \in TPL,\ \alpha\ \overline{T}{\leq}\ ;\ \overline{T}{\in}\ \Phi \text{ iff } \alpha\ \overline{T}{\in}\ T{\downarrow}_X(\Phi).$$

Now we see that

$$\begin{aligned} \alpha \in {\downarrow}_{TX}\lambda^T(\Phi) &\Leftrightarrow \alpha\ \overline{T}{\leq}\ ;\ \overline{T}{\in}\ \Phi && \text{by def. of } {\downarrow} \text{ and } \lambda^T, \\ &\Leftrightarrow \alpha\ \overline{T}{\in}\ T{\downarrow}_X(\Phi) && \text{by the above,} \\ &\Leftrightarrow \alpha \in \lambda^T\,(T{\downarrow}_X(\Phi)) && \text{by def. of } \lambda^T. \end{aligned}$$

(2). For all $a, b \in X$, we have $b \leq a$ iff $b \in {\downarrow}_X\{a\}$. It follows by relation lifting that

$$\forall \alpha, \beta \in TX,\ \beta\ \overline{T}{\leq}\ \alpha \text{ iff } \beta\ \overline{T}{\in}\ T{\downarrow}_X \circ T\eta_X(\alpha).$$

It now follows by an argument analogous to that for (1) above that (2) holds.

(3). Observe that for all $A, A' \in PX$ and all $a \in X$, we have that $a \in A' \subseteq A$ implies that $a \in A$. The statement follows by relation lifting. □

We are now ready to prove that $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is indeed a flat site.

Lemma 3.25. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor. If $\mathbb{L}$ is a frame then $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is a flat site.*

Proof. We have already know from Lemma 2.7 that $\langle TL, \overline{T}{\leq}\rangle$ is a pre-order, so what remains to be shown is that the relation $\lhd_0^{\mathbb{L}}$ is compatible with the pre-order. Fix $\alpha \in TL$ and $\Phi \in TPL$ such that $\alpha\ \overline{T}{\leq}\ T\bigvee(\Phi)$, so that $\alpha \lhd_0^{\mathbb{L}} \lambda^T(\Phi)$. We need to show that

$$\forall \beta \in TL,\ \text{if } \beta\ \overline{T}{\leq}\ \alpha \text{ then } \exists \Gamma \in TPL \text{ with } \Gamma \subseteq {\downarrow}_{TL}\{\beta\} \cap {\downarrow}_{TL}\lambda^T(\Phi) \text{ and } \beta \lhd_0^{\mathbb{L}} \Gamma. \tag{34}$$

But this is easy to see: if $\beta\ \overline{T}{\leq}\ \alpha$ then since $\alpha\ \overline{T}{\leq}\ T\bigvee(\Phi)$, it follows by transitivity of $\overline{T}{\leq}$ that $\beta\ \overline{T}{\leq}\ T\bigvee(\Phi)$. Now by Lemma 3.23 there exists $\Phi' \in TPL$ such that $\alpha\ \overline{T}{\leq}\ T\bigvee(\Phi')$, $\Phi'\ \overline{T}{\subseteq}\ T{\downarrow}_L \circ T\eta(\beta)$ and $\Phi'\ \overline{T}{\subseteq}\ T{\downarrow}_L\Phi$. Define $\Gamma := \lambda^T(\Phi')$, then we have $\beta \lhd_0 \Gamma$ by definition of $\lhd_0^{\mathbb{L}}$ that; moreover, it now follows from Lemma 3.24 that $\Gamma \subseteq {\downarrow}_{TL}\{\beta\} \cap {\downarrow}_{TL}\lambda^T(\Phi)$. We conclude that (34) holds. Since $\alpha \in TL$ and $\Phi \in TPL$ were arbitrary, we have shown that $\lhd_0^{\mathbb{L}}$ is compatible with the order $\overline{T}{\leq}$, so that $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is a flat site. □

Having established that $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$ is a flat site, we will now prove that it presents $V_T\mathbb{L}$, i.e. that $V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$.

Theorem 3.26. *Let $\mathbb{L}$ be a frame and let T be a standard, finitary, weak pullback-preserving functor. Then $V_T\mathbb{L}$ admits the following flat site presentation:*

$$V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle,$$

where $\lhd_0^{\mathbb{L}} = \{(\alpha, \lambda^T(\Phi)) \in L \times PTL \mid \alpha \;\overline{T}\leq T\bigvee(\Phi)\}$, and in each direction, the isomorphism is the unique frame homomorphism extending the identity map id_{TL} on the set of generators of $V_T\mathbb{L}$ and $\mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle$, respectively.

Proof. For this proof, we denote the insertion of generators from TL to $V_T\mathbb{L}$ by ∇, and from TL to $\mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle$ by $\heartsuit$. We will show that

(1) the function $\heartsuit\colon TL \to \mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle$ is compatible with the relations (∇1), ($\nabla 2'$) and (∇3), and
(2) that the function $\nabla\colon TL \to V_T\mathbb{L}$ has the following properties:
  (a) ∇ is order-preserving;
  (b) $1 = \bigvee\{\nabla\alpha \mid \alpha \in TL\}$;
  (c) for all $\alpha, \beta \in TL$, $\nabla\alpha \wedge \nabla\beta = \bigwedge\{\nabla\gamma \mid \delta \;\overline{T}\leq \alpha, \beta\}$;
  (d) for all $\alpha \lhd_0^{\mathbb{L}} \Gamma$, $\nabla\alpha \leq \bigvee\{\nabla\beta \mid \beta \in \Gamma\}$.

(1). First consider (∇1). Suppose that $\alpha, \beta \in TL$ such that $\alpha \;\overline{T}\leq\; \beta$; we have to show that $\heartsuit\alpha \leq \heartsuit\beta$. This follows immediately from the fact that $\heartsuit\colon TL \to \mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle$ is order-preserving. Secondly, consider ($\nabla 2'$). Let $\Gamma \in P_\omega TL$, we then have to show that

$$\bigwedge\nolimits_{\gamma\in\Gamma}\heartsuit\gamma \leq \bigvee\{\heartsuit\delta \mid \forall\gamma \in \Gamma,\, \delta \;\overline{T}\leq \gamma\}. \tag{35}$$

Recall from §2.4 that since $\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle$ is a flat site, we know that $1 = \bigvee\{\heartsuit\alpha \mid \alpha \in TL\}$ and that for all $\alpha, \beta \in TL$, $\heartsuit\alpha \wedge \heartsuit\beta = \bigwedge\{\heartsuit\gamma \mid \delta \;\overline{T}\leq \alpha, \beta\}$. It now follows by induction on the size of Γ that (35) holds.

Finally for (∇3), take $\Phi \in TPL$. We have to show that $\heartsuit T\bigvee(\Phi) \leq \bigvee\{\heartsuit\beta \mid \beta \in \lambda^T(\Phi)\}$. This follows immediately from the definition of $\lhd_0^{\mathbb{L}}$, since $T\bigvee(\Phi) \;\overline{T}\leq T\bigvee(\Phi)$. We conclude that $\heartsuit\colon TL \to \mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle$ is compatible with the relations (∇1), (∇2) and (∇3) and thus there must be a unique frame homomorphism $f\colon V_T\mathbb{L} \to \mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle$ which extends $\heartsuit$.

(2). We first have to show that ∇ is order-preserving, i.e. that if $\alpha \;\overline{T}\leq\; \beta$, then $\nabla\alpha \leq \nabla\beta$. This follows immediately from (∇1). Secondly, we must show that (2)(b) and (2)(c) are satisfied, but this follows immediately from ($\nabla 2'$). Finally, consider (2)(d), i.e. suppose that $\alpha \lhd_0^{\mathbb{L}} \Gamma$. By definition of $\lhd_0^{\mathbb{L}}$, there is some $\Phi \in TPL$ such that $\alpha \;\overline{T}\leq T\bigvee(\Phi)$ and $\lambda^T(\Phi) = \Gamma$. Now we need to show that $\nabla\alpha \leq \bigvee\{\nabla\beta \mid \beta \in \lambda^T(\Phi)\}$. This is easy to see, since

$$\begin{aligned}
\nabla\alpha &\leq \nabla T\textstyle\bigvee(\Phi) && \text{by } (\nabla 1),\\
&\leq \textstyle\bigvee\{\nabla\beta \mid \beta \in \lambda^T(\Phi)\} && \text{by } (\nabla 3).
\end{aligned}$$

It follows that (2)(d) holds; consequently, there exists a unique frame homomorphism

$$g\colon\ \mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle \to V_T\mathbb{L},$$

extending ∇.

Finally, it is easy to see that

$$gf = id_{V_T\mathbb{L}} \text{ and } fg = id_{\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle},$$

so that indeed $V_T\mathbb{L} \simeq \mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle$. □

In light of Theorem 3.26 above, we denote the insertion of generators by $\nabla\colon TL \to \mathrm{Fr}\langle TL, \overline{T}\leq, \lhd_0^{\mathbb{L}}\rangle$. We now arrive at the most important corollary of Theorem 3.26, which says that every element of $V_T\mathbb{L}$ has a disjunctive normal form.

Corollary 3.27. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor and let $\mathbb{L}$ be a frame. Then for all $x \in V_T\mathbb{L}$, there is a $\Gamma \in PTL$ such that $x = \bigvee\{\nabla\gamma \mid \gamma \in \Gamma\}$.*

Proof. By Theorem 3.26 we know that $V_T\mathbb{L} \simeq \mathrm{SupLat}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$. The statement now follows by Fact 2.13. □

Remark 3.28. It is not hard to show that

$$\mathrm{SupLat}\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle \simeq \mathrm{SupLat}\langle TL \mid (\nabla 1), (\nabla 3)\rangle.$$

Consequently, by Theorem 3.26 and Fact 2.13, the *order* on $V_T\mathbb{L}$ is uniquely determined by the relations $(\nabla 1)$ and $(\nabla 3)$.

4. Preservation results

Now that we have established the T-powerlocale construction, we can set about to prove that it is well-behaved. One particular kind of good behavior is to ask that it preserves algebraic properties. In this section, we present several initial results in this area. We start by briefly reviewing some of the preservation properties of V, the usual Vietoris powerlocale, in §4.1; in addition, we prove that V preserves compactness. In §4.2, we show that V_T, the T-powerlocale construction, preserves regularity and zero-dimensionality. Finally, in §4.3 we show that if we assume that T maps finite sets to finite sets, then V_T preserves the combination of compactness and zero-dimensionality.

4.1. Preservation properties of V.

There are various relations between properties of $\mathbb{L}$ and of $V\mathbb{L}$. For instance, (Johnstone, 1985) shows that $\mathbb{L}$ is regular, completely regular, zero-dimensional or compact regular iff $V\mathbb{L}$ is, and also that if $\mathbb{L}$ is locally compact then so is $V\mathbb{L}$. The same paper also mentions without proof that if $\mathbb{L}$ is compact then so is $V\mathbb{L}$, referring to a proof by transfinite induction similar to that used for the localic Tychonoff theorem in (Johnstone, 1982). The paper leaves open the converse question, of whether $V\mathbb{L}$ compact implies so is $\mathbb{L}$. We shall give here a constructive (topos-valid) proof using preframe techniques that $\mathbb{L}$ is compact iff $V\mathbb{L}$ is.

Definition 4.1. A frame $\mathbb{L}$ (or, more properly, its locale) is *compact* if whenever $1 \leq \bigvee_i^{\uparrow} a_i$ then $1 \leq a_i$ for some i.

The following constructive proof is a routine application of the techniques in (Johnstone & Vickers, 1991).

Theorem 4.2. *$\mathbb{L}$ is compact iff $V\mathbb{L}$ is.*

Proof. $\Rightarrow$: $\mathbb{L}$ is compact iff the function $\mathbb{L} \to \Omega$ that maps $a \in \mathbb{L}$ to the truth value of $a = 1$ is a preframe homomorphism, i.e. preserves finite meets and directed joins. This function is characterized by being right adjoint to the unique frame homomorphism $!\colon \Omega \to \mathbb{L}$ and so to prove compactness it suffices to define a preframe homomorphism $\mathbb{L} \to \Omega$ and show that it is right adjoint to !. If $\mathbb{L}$ is presented – as a *frame* – by generators and relations, then the "preframe coverage theorem" of (Johnstone & Vickers, 1991) shows how to derive a presentation as preframe, which can then be used for defining preframe homomorphisms from $\mathbb{L}$. The strategy is to generate a $\vee$-semilattice from the generators, and add relations to ensure a $\vee$-*stability* condition analogous to the $\wedge$-stability used in Johnstone's coverage theorem (Johnstone, 1982).

Our first step is to apply the preframe coverage theorem to derive a preframe presentation of $V\mathbb{L}$. We show

$$\begin{array}{lll} V\mathbb{L} \cong \mathrm{Fr}\langle P_\omega\mathbb{L}\times\mathbb{L} & (\text{qua } \vee\text{-semilattice}) \mid & \\ & 1 \leq (\gamma\cup\{1\}, d) & \\ & (\gamma\cup\{a\}, d)\wedge(\gamma\cup\{b\}, d) \leq (\gamma\cup\{a\wedge b\}, d) & \\ & (\gamma\cup\{\bigvee^\uparrow A\}, d) \leq \bigvee^\uparrow_{a\in A}(\gamma\cup\{a\}, d) & (A \text{ directed}) \\ & (\gamma, \bigvee^\uparrow A\vee d) \leq \bigvee^\uparrow_{a\in A}(\gamma, a\vee d) & (A \text{ directed}) \\ & (\gamma\cup\{a\}, d)\wedge(\gamma, b\vee d) \leq (\gamma, (a\wedge b)\vee d) & \\ & (\gamma\cup\{a\vee b\}, d) \leq (\gamma\cup\{a\}, b\vee d) & \\ & \rangle. & \end{array}$$

The $\vee$-semilattice structure on $P_\omega\mathbb{L}\times\mathbb{L}$ is the product structure from $\cup$ on $P_\omega\mathbb{L}$ and $\vee$ on $\mathbb{L}$. The homomorphisms between the frame presented above and $V\mathbb{L}$ are given by

$$\Box a \mapsto (\{a\}, 0),\ \Diamond a \mapsto (\emptyset, a)$$
$$(\gamma, d) \mapsto \bigvee_{c\in\gamma}\Box c\vee\Diamond d.$$

The relations shown are $\vee$-stable, so the preframe coverage shows that

$$V\mathbb{L} \cong \mathrm{PreFr}\langle P_\omega\mathbb{L}\times\mathbb{L}\ (\text{qua poset}) \mid \text{same relations as above}\ \rangle.$$

We can now define a preframe homomorphism $\varphi\colon V\mathbb{L}\to\Omega$ by

$$\varphi(\gamma, d) = \exists c\in\gamma.\, c\vee d = 1.$$

To motivate this, we want criteria for $\bigvee_{c\in\gamma}\Box c\vee\Diamond d = 1$, and intuitively this means that for every sublocale K corresponding to a point of $V\mathbb{L}$ either K is included in some $c\in\gamma$ or K meets d. Taking K to be the closed complement of d, we get the given condition. This is not a rigorous argument, since that closed complement is not necessarily a point of $V\mathbb{L}$. However, the rest of our argument validates the choice. The relations in the preframe presentation of $V\mathbb{L}$ are largely easy to check. We shall just mention the penultimate one. Suppose $(\gamma\cup\{a\}, d)$ and $(\gamma, b\vee d)$ are both mapped to 1. We have either some $c\in\gamma$ with $c\vee d = 1$, in which case $c\vee(a\wedge b)\vee d = 1$, or we have $a\vee d = 1$ and in addition some $c'\in\gamma$ with $c'\vee b\vee d = 1$. In this latter case $c'\vee(a\wedge b)\vee d = 1$.

Next we show that φ is right adjoint to $!\colon\Omega\to V\mathbb{L}$, the unique frame homomorphism defined by

$$!(p) = \bigvee\{1\mid p\} = \bigvee\nolimits^\uparrow(\{0\}\cup\{1\mid p\}).$$

We must show $\varphi(!(p))\geq p$ and $!(\varphi(\gamma, d))\leq(\gamma, d)$.

$$\begin{aligned} \varphi(!(p)) &= \varphi\left(\bigvee\nolimits^\uparrow(\{0\}\cup\{1\mid p\})\right) \\ &= \varphi(\emptyset, 0)\vee\bigvee\{\varphi(\{1\}, 0)\mid p\}\geq p \end{aligned}$$

since if p holds then the disjuncts include $\varphi(\{1\}, 0) = 1$. For the other inequality, we must show that

$$\bigvee\{1\mid\varphi(\gamma, d)\}\leq(\gamma, d).$$

If $\varphi(\gamma, d)$ holds true then $c\vee d = 1$ for some $c\in\gamma$, so

$$1 = (\{1\}, 0) = (\{c\vee d\}, 0)\leq(\{c\}, d)\leq(\gamma, d).$$

$\Leftarrow$: Suppose in $\mathbb{L}$ we have $1 = \bigvee^\uparrow_i a_i$. Then in $V\mathbb{L}$ we have $1 = \Box 1 = \bigvee^\uparrow_i\Box a_i$ and so $1 = \Box a_i$ for some i. Applying i to both sides gives $1 = a_i$. $\Box$

4.2. **Regularity and zero-dimensionality.** The purpose of this subsection is to prove that the operation V_T preserves regularity and zero-dimensionality of frames. Both of these notions are defined in terms of the well-inside relation $\lessdot$; accordingly, the main technical result of this subsection states that if $\alpha\ \overline{T}\!\lessdot\ \beta$, then also $\nabla\alpha \lessdot_{V_T\mathbb{L}} \nabla\beta$. We first recall some notions leading up to the definition of regularity.

Definition 4.3. Given two elements a, b of a distributive lattice $\mathbb{L}$, we say that a is *well inside* b, notation: $a \lessdot b$, if there is some c in $\mathbb{L}$ such that $a \wedge c = 0$ and $b \vee c = 1$. If $a \lessdot a$ we say a is *clopen*. We denote the clopen elements of $\mathbb{L}$ by $C_{\mathbb{L}}$.

In case $\mathbb{L}$ is a frame, in the definition of $\lessdot$, for the element c witnessing that $a \lessdot b$ we may always take the Heyting complementation $\neg a$ of a. In other words, $a \lessdot b$ iff $b \vee \neg a = 1$. Consequently, if a is clopen then $a \vee \neg a = 1$. In the sequel we will use not only this fact, but also the following properties of $\lessdot$ without warning. For proofs, see (Johnstone, 1982, §III-1.1).

Fact 4.4. *Let $\mathbb{L}$ be a frame.*
1. $\lessdot \subseteq \leq$;
2. $\leq ; \lessdot ; \leq \subseteq \lessdot$;
3. *for $X \in P_\omega L$, if $\forall x \in X. x \lessdot y$ then $\bigvee X \lessdot y$;*
4. *for $X \in P_\omega L$, if $\forall x \in X. y \lessdot x$ then $y \lessdot \bigwedge X$;*
5. *$a \lessdot a$ iff a has a complement.*

Definition 4.5. A frame $\mathbb{L}$ is *regular* if every $a \in \mathbb{L}$ satisfies
$$a = \bigvee\{b \in L \mid b \lessdot a\}.$$
We say $\mathbb{L}$ is *zero-dimensional* if for all $a \in \mathbb{L}$,
$$a = \bigvee\{b \in C_{\mathbb{L}} \mid b \leq a\}.$$

We record the following useful property of $C_{\mathbb{L}}$ (see Johnstone, 1982, §III-1.1):

Fact 4.6. *Let $\mathbb{L}$ be a frame. Then $\langle C_{\mathbb{L}}, \wedge, \vee, 0, 1\rangle$ is a sublattice of $\mathbb{L}$.*

We define a function $\Downarrow\colon PL \to PC_{\mathbb{L}}$ which maps $A \in PL$ to $\downarrow A \cap C_{\mathbb{L}}$.

Lemma 4.7. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor. If $\mathbb{L}$ is a zero-dimensional frame, then*
1. $\forall \alpha \in TL$, $\nabla\alpha = \bigvee\{\nabla\beta \mid \beta \in TC_{\mathbb{L}},\ \beta\ \overline{T}\!\leq \alpha\}$;
2. $\forall \Phi \in TPL$, $T\bigvee(\Phi) = T\bigvee \circ T\Downarrow(\Phi)$;
3. $\forall \Phi \in TPL$, $\forall \alpha \in TL$, $[\alpha \in TC_{\mathbb{L}}$ *and* $\alpha\ \overline{T}\!\leq ; \overline{T}\!\in \Phi]$ *iff* $\alpha \in \lambda^T\,(T\Downarrow(\Phi))$.

Similarly to (1), if $\mathbb{L}$ is regular then $\forall \alpha \in TL$, $\nabla\alpha = \bigvee\{\nabla\beta \mid \beta \in TL,\ \beta\ \overline{T}\!\lessdot \alpha\}$.

Proof. (1). First, observe that for all $a \in L$, we have that
$$\begin{aligned} a &= \bigvee\{b \in C_{\mathbb{L}} \mid b \leq a\} && \text{by zero-dimensionality,}\\ &= \bigvee \Downarrow\{a\} && \text{by definition of } \Downarrow,\\ &= \bigvee \Downarrow \circ \eta(a) && \text{by def. of } \eta\colon \mathrm{Id}_{\mathsf{Set}} \to P. \end{aligned}$$
By relation lifting, it follows that
$$\forall \alpha \in TL,\ \alpha = T\bigvee \circ T\Downarrow \circ T\eta(\alpha). \tag{36}$$
Now observe that for all $a, b \in L$, we have $b \in \Downarrow\eta(a)$ iff $b \in C_{\mathbb{L}}$ and $b \leq a$. By relation lifting, it follows that
$$\forall \alpha, \beta \in TL,\ \left[\beta\ \overline{T}\!\in T\Downarrow \circ T\eta(\alpha) \text{ iff } \beta \in TC_{\mathbb{L}} \text{ and } \beta\ \overline{T}\!\leq \alpha\right]. \tag{37}$$

Combining these two observations, we see that

$$\begin{aligned}
\nabla\alpha &= \nabla\,(T\bigvee \circ T\Downarrow\circ T\eta(\alpha)) && \text{by (36)},\\
&= \bigvee\{\nabla\beta \mid \beta\,\overline{T}{\in}\,T\Downarrow\circ T\eta(\alpha)\} && \text{by } (\nabla 3),\\
&= \bigvee\{\nabla\beta \mid \beta\in TC_{\mathbb{L}},\ \beta\,\overline{T}{\leq}\,\alpha\} && \text{by (37)}.
\end{aligned}$$

(2). It follows by zero-dimensionality of $\mathbb{L}$ that for all $A\in PL$, we have $\bigvee A=\bigvee\Downarrow A$. Consequently, by relation lifting, (2) holds.

(3). Take $a\in L$ and $A\in PL$. Then

$$\begin{aligned}
a\in\Downarrow A &\Leftrightarrow a\in C_{\mathbb{L}} \text{ and } \exists b\in A,\ a\leq b && \text{by definition of } \Downarrow,\\
&\Leftrightarrow a\in C_{\mathbb{L}} \text{ and } a\leq;\in A && \text{by def. of relation composition.}
\end{aligned}$$

It follows by relation lifting that

$$\forall\Phi\in TPL, \forall\alpha\in TL,\ \alpha\,\overline{T}{\in}\,T\Downarrow(\Phi) \text{ iff } \alpha\in TC_{\mathbb{L}} \text{ and } \alpha\,T{\leq};T{\in}\,\Phi.$$

Now it follows by definition of $\lambda^T(\Phi)$ that (3) holds.

For the last part of the proof, first observe that if $\mathbb{L}$ is regular, then for all $a\in L$, $a=\bigvee w(a)$, where we temporarily define $w\colon L\to PL$ as

$$w\colon a\mapsto\{b\in L\mid b\lessdot a\}.$$

By relation lifting, it follows that

$$T\bigvee\circ Tw = id_L. \tag{38}$$

Moreover, it follows by definition of $w\colon L\to PL$ that for all $a,b\in L$, $b\in w(a)$ iff $b\lessdot a$. Consequently,

$$\forall\alpha,\beta\in TL,\ \beta\,\overline{T}{\in}\,Tw(\alpha) \text{ iff } \beta\,\overline{T}{\lessdot}\,\alpha. \tag{39}$$

Now we see that for any $\alpha\in TL$,

$$\begin{aligned}
\nabla\alpha &= \nabla\,(T\bigvee\circ Tw(\alpha)) && \text{by (38)},\\
&= \bigvee\{\nabla\beta\mid\beta\,\overline{T}{\in}\,Tw(\alpha)\} && \text{by } (\nabla 3),\\
&= \bigvee\{\nabla\beta\mid\beta\,\overline{T}{\lessdot}\,\alpha\} && \text{by (39)}.
\end{aligned}$$

□

The key technical lemma of this subsection states that relation lifting preserves the $\lessdot$-relation.

Lemma 4.8. *Let T be a standard, finitary, weak pullback-preserving functor and let $\mathbb{L}$ be a frame. Then*

$$\textit{for all } \alpha,\beta\in TL:\ \alpha\,\overline{T}{\lessdot}\,\beta \textit{ implies } \nabla\alpha\lessdot_{V_T\mathbb{L}}\nabla\beta. \tag{40}$$

Proof. Let $\alpha,\beta\in TL$ be such that $\alpha\,\overline{T}{\lessdot}\,\beta$. Our aim will be to show that $\nabla\alpha\lessdot_{V_T\mathbb{L}}\nabla\beta$.

We may assume without loss of generality that

$$\beta=(Tf)\alpha \text{ for some } f\colon Base^T(\alpha)\to Base^T(\beta) \text{ such that } a\lessdot fa \text{ for all } a\in Base^T(\alpha). \tag{41}$$

To justify this assumption, assume that we have a proof of (40) for all β satisfying (41). To derive (40) in the general case, consider arbitrary elements $\alpha,\beta'\in TL$ such that $\alpha\,\overline{T}{\lessdot}\,\beta'$. In order to show that $\nabla\alpha\,\overline{T}{\lessdot}\,\nabla\beta'$, consider the map $f\colon Base^T(\alpha)\to L$ given by $f(a):=\bigwedge\{b\in Base^T(\beta')\mid a\lessdot b\}$. On the basis of Fact 4.4 it is not difficult to see that $Gr\,(f)\subseteq\lessdot$ and so by the properties of relation lifting we obtain $Gr\,(Tf)\subseteq\overline{T}{\lessdot}$. In particular, we find that $\alpha\,\overline{T}{\lessdot}\,(Tf)\alpha$; thus by our assumption we may conclude that $\nabla\alpha\lessdot\nabla(Tf)\alpha$. Also, observe that $a\lessdot b$ implies $fa\leq b$, for all $a\in Base^T(\alpha)$ and $b\in Base^T(\beta')$. Hence by Lemma 2.7 we may conclude from $\alpha\,\overline{T}{\lessdot}\,\beta'$ that $(Tf)\alpha\,\overline{T}{\leq}\,\beta'$, which gives

$\nabla(Tf)\alpha \leq \nabla\beta'$. Combining our observations thus far, by Fact 4.4 it follows from $\nabla\alpha \leqslant \nabla(Tf)\alpha$ and $\nabla(Tf)\alpha \leq \nabla\beta'$ that $\nabla\alpha \leqslant \nabla\beta'$ indeed. Thus our assumption (41) is justified indeed.

Turning to the proof itself, consider the map $h\colon P\,Base^T(\alpha) \to L$ given by

$$h(A) := \bigwedge\left(\{\neg a \mid a \in A\} \cup \{fa \mid a \notin A\}\right).$$

Our first observation is that, since by assumption $\neg a \vee fa = 1_{\mathbb{L}}$ for each $a \in Base^T(\alpha)$, we may infer that

$$1_{\mathbb{L}} = \bigwedge\{\neg a \vee fa \mid a \in Base^T(\alpha)\},$$

a straightforward application of the (finitary) distributive law yields that

$$1_{\mathbb{L}} = \bigvee\{h(A) \mid A \in P\,Base^T(\alpha)\}. \tag{42}$$

Define $X \subseteq L$ to be the range of h, so that we may think of h as a surjection $h\colon P\,Base^T(\alpha) \to X$, and read (42) as saying that $1 = \bigvee X$. Using Lemma 3.10(5), from the latter observation we may infer that

$$1_{V_T\mathbb{L}} = \bigvee\{\nabla\xi \mid \xi \in TX\}. \tag{43}$$

However, from $h\colon P\,Base^T(\alpha) \to X$ being surjective we may infer that $Th\colon TP\,Base^T(\alpha) \to TX$ is also surjective, so that we may read (43) as

$$1_{V_T\mathbb{L}} = \bigvee\{\nabla Th(\Phi) \mid \Phi \in TP\,Base^T(\alpha)\}. \tag{44}$$

This leads us to the key observation in our proof: We may partition the set $\{Th(\Phi) \mid \Phi \in TP\,Base^T(\alpha)\}$ into elements γ such that $\nabla\gamma \leq \nabla\beta$, and elements γ satisfying $\nabla\alpha \wedge \nabla\gamma = 0_{V_T\mathbb{L}}$.

Claim 1. Let $\Phi \in TP\,Base^T(\alpha)$.
(a) If $(\alpha, \Phi) \in \overline{T}\notin$, then $Th(\Phi)\,\overline{T}\leq \beta$;
(b) if $(\alpha, \Phi) \notin \overline{T}\notin$, then $\nabla\alpha \wedge \nabla Th(\Phi) = 0_{V_T\mathbb{L}}$.

Proof of Claim For part (a), it is not hard to see that

$$a \notin A \Rightarrow h(A) \leq f(a), \text{ for all } a \in Base^T(\alpha), A \in P\,Base^T(\alpha).$$

From this it follows by Lemma 2.7 that

$$\alpha\,\overline{T} \notin \Phi \Rightarrow Th(\Phi)\,\overline{T}\leq (Tf)(\alpha) = \beta.$$

For part (b), assume that $\nabla\alpha \wedge \nabla Th(\Phi) > 0_{V_T\mathbb{L}}$. It suffices to derive from this that $\alpha\,\overline{T}\notin \Phi$.

Let $\leq'$ be the restriction of $\leq$ to the non-zero part of $\mathbb{L}$, that is, $\leq' := \leq\restriction_{L'\times L'}$, where $L' = L \setminus \{0_{\mathbb{L}}\}$. We claim that for all $\gamma, \delta \in TL$:

$$\nabla\gamma \wedge \nabla\delta > 0_{V_T\mathbb{L}} \;\Rightarrow\; (\gamma, \delta) \in \overline{T}\geq' ; \overline{T}\leq'. \tag{45}$$

To see this, assume that $\nabla\gamma \wedge \nabla\delta > 0_{V_T\mathbb{L}}$, and observe that Lemma 3.5 yields the existence of a $\theta \in TL$ such that $\nabla\theta > 0_{V_T\mathbb{L}}$ and $\theta\,\overline{T}\leq \gamma, \delta$. It follows from Lemma 3.10(1) that γ, δ and θ all belong to TL', and so θ is witnesses to the fact that $(\gamma, \delta) \in \overline{T}\geq' ; \overline{T}\leq'$.

By (45) and the assumption on α and Φ it follows that $(\alpha, \Phi) \in \overline{T}\geq' ; \overline{T}\leq' ; (Gr\,Th)^{\smile}$, and so by Fact 2.6 we obtain

$$(\alpha, \Phi) \in \overline{T}(\geq' ; \leq' ; (Gr\,h)^{\smile}) \tag{46}$$

The crucial observation now is that

$$\geq' ; \leq' ; (Gr\,h)^{\smile} \;\subseteq\; \notin. \tag{47}$$

For a proof, take a pair $(a, A) \in L \times PL$ in the LHS of (47), and suppose for contradiction that $a \in A$. Then by definition of h we obtain $h(A) \leq \neg a$, so that $a \wedge h(A) = 0_{\mathbb{L}}$. But if $a \geq' ; \leq' ; (Gr\,h)^{\smile} A$, then there must be some b such that $b \leq' a, h(A)$, and by definition of $\leq'$ this can only be the case if $b > 0_{\mathbb{L}}$. This gives the desired contradiction.

Finally, by monotonicity of relation lifting, it is an immediate consequence of (46) and (47) that $\alpha \, \overline{T}{\not\in}\, \Phi$. This finishes the proof of the Claim.

On the basis of the Claim it is straightforward to finish the proof. Define

$$c := \bigvee \left\{ Th(\Phi) \mid \Phi \in TP\,Base^T(\alpha) \text{ such that } (\alpha, \Phi) \notin \overline{T}{\not\in} \right\},$$

then we may calculate that

$$\begin{aligned}
& c \vee \nabla\beta \\
& \geq c \vee \bigvee \left\{ Th(\Phi) \mid \Phi \in TP\,Base^T(\alpha) \text{ such that } (\alpha, \Phi) \in \overline{T}{\not\in} \right\} && \text{(Claim 1(a))} \\
& = \bigvee \left\{ Th(\Phi) \mid \Phi \in TP\,Base^T(\alpha) \right\} && \text{(definition of } c\text{)} \\
& = 1_{V_T\mathbb{L}} && \text{(equation (44))}
\end{aligned}$$

and

$$\begin{aligned}
& \nabla\alpha \wedge c \\
& = \bigvee \left\{ \nabla\alpha \wedge Th(\Phi) \mid \Phi \in TP\,Base^T(\alpha) \text{ such that } (\alpha, \Phi) \notin \overline{T}{\not\in} \right\} && \text{(distributivity)} \\
& = \bigvee \left\{ 0_{V_T\mathbb{L}} \mid \Phi \in TP\,Base^T(\alpha) \text{ such that } (\alpha, \Phi) \notin \overline{T}{\not\in} \right\} && \text{(Claim 1(b))} \\
& = 0_{V_T\mathbb{L}}
\end{aligned}$$

In other words, c witnesses that $\nabla\alpha \preccurlyeq_{V_T\mathbb{L}} \nabla\beta$. □

We now arrive at the main result of this subsection, namely, that the T-powerlocale construction preserves regularity and zero-dimensionality.

Theorem 4.9. *Let $\mathbb{L}$ be a frame and let T be a standard, finitary, weak pullback-preserving functor.*

(1) *If $\mathbb{L}$ is regular then so is $V_T\mathbb{L}$.*
(2) *If $\mathbb{L}$ is zero-dimensional then so is $V_T\mathbb{L}$.*

Proof. (1). By Corollary 3.27, it suffices to show that for all $\alpha \in TL$,

$$\nabla\alpha = \bigvee\{\nabla\beta \in V_T\mathbb{L} \mid \nabla\beta \preccurlyeq \nabla\alpha\}. \tag{48}$$

Take $\alpha \in TL$; we see that

$$\begin{aligned}
\nabla\alpha &= \bigvee\{\nabla\beta \mid \beta \, \overline{T}{\preccurlyeq}\, \alpha\} && \text{by Lemma 4.7,} \\
&\leq \bigvee\{\nabla\beta \mid \nabla\beta \preccurlyeq_{V_T\mathbb{L}} \nabla\alpha\} && \text{by Lemma 4.8,} \\
&\leq \nabla\alpha && \text{since } {\preccurlyeq} \subseteq {\leq}.
\end{aligned}$$

It follows that (48) holds, concluding the proof of part (1).

(2). Again by Corollary 3.27, it suffices to show that for all $\alpha \in TL$,

$$\nabla\alpha = \bigvee\{\nabla\beta \mid \nabla\beta \in C_{V_T\mathbb{L}}, \nabla\beta \leq \nabla\alpha\}. \tag{49}$$

The main observation here is that

$$\forall\beta \in TC_{\mathbb{L}}, \nabla\beta \in C_{V_T\mathbb{L}}. \tag{50}$$

To see why, recall that $C_{\mathbb{L}} := \{b \in L \mid b \preccurlyeq b\}$, so that for all $b \in C_{\mathbb{L}}$, $b = b$ implies $b \preccurlyeq b$. Consequently, by relation lifting,

$$\forall\beta \in TC_{\mathbb{L}}, \beta \, \overline{T}{\preccurlyeq}\, \beta.$$

It follows by Lemma 4.8 that (50) holds. Now

$$\begin{aligned}
\nabla\alpha &= \bigvee\{\nabla\beta \mid \beta \in TC_{\mathbb{L}},\ \beta\ \overline{T}{\leq}\ \alpha\} && \text{by Lemma Lemma 4.7(1),}\\
&\leq \bigvee\{\nabla\beta \in C_{V_T\mathbb{L}} \mid \beta\ \overline{T}{\leq}\ \alpha\} && \text{by (50),}\\
&\leq \bigvee\{\nabla\beta \in C_{V_T\mathbb{L}} \mid \nabla\beta \leq \nabla\alpha\} && \text{by } (\nabla 1),\\
&= \nabla\alpha && \text{by order theory.}
\end{aligned}$$

It now follows that (49) holds; consequently we see that (2) holds. □

4.3. **Compactness + zero-dimensionality.** In this subsection, we will show that if $\mathbb{L}$ is compact and zero-dimensional, then so is $V_T\mathbb{L}$. Our proof strategy is as follows. Given a compact zero-dimensional frame $\mathbb{L}$, we will define a new construction $V_T^C\mathbb{L}$ which is guaranteed to be compact, and then we show that $V_T\mathbb{L} \simeq V_T^C\mathbb{L}$.

We define a flat site presentation $\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$, where

$$\lhd_0^C := \{(\alpha, \lambda^T(\Phi)) \in TC_{\mathbb{L}} \times PTL \mid \alpha\ \overline{T}{\leq}\ T\bigvee(\Phi),\ \Phi \in TP_\omega C_{\mathbb{L}}\}.$$

Observe that we view $TC_{\mathbb{L}}$ as a substructure of TL, which is justified by the fact that $C_{\mathbb{L}}$ is a sublattice of $\mathbb{L}$ (Fact 4.6): this fact tells us that $\bigvee\colon PL \to L$ restricts to a function from $P_\omega C_{\mathbb{L}}$ to $C_{\mathbb{L}}$; consequently, $T\bigvee$ maps $TP_\omega C_{\mathbb{L}}$ to $TC_{\mathbb{L}}$, by standardness of T. Below, we will need the following property of relation lifting with respect to ordered sets.

Lemma 4.10. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor and let $\mathbb{P}$ be a poset with a top element 1. Then for every $\beta \in TP$ there is some $\alpha \in T\{1\}$ such that $\beta\ \overline{T}{\leq}\ \alpha$;*

Proof. Consider the following function at the ground level: $f\colon P \to \{1\}$, where f is the constant function $f\colon b \mapsto 1$. Then for all $b \in P$, we have $b \leq f(b)$ and $f(b) \in \{1\}$. By relation lifting, we see that for all $\beta \in TP$, $\beta\ \overline{T}{\leq}\ Tf(\beta)$ and $Tf(\beta) \in T\{1\}$. The statement follows. □

Lemma 4.11. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor and let $\mathbb{L}$ be a frame. Then $\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ is a flat site. Moreover, if T maps finite sets to finite sets then $\mathrm{Fr}\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ is a compact frame.*

Proof. Because $C_{\mathbb{L}}$ is a meet-subsemilattice of $\mathbb{L}$, we can apply Lemma 3.23 to $TC_{\mathbb{L}}$. Now the proof that $\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ is a flat site is analogous to that of Lemma 3.25.

Now suppose that T maps finite sets to finite sets. Then for all $\Phi \in TP_\omega C_{\mathbb{L}}$, it follows by Fact 2.11(3) that $\lambda^T(\Phi)$ is finite. Consequently,

$$\forall\alpha \lhd_0^C \lambda^T(\Phi),\ \lambda^T(\Phi) \text{ is finite.}$$

Moreover, by Lemma 4.10,

$$TC_{\mathbb{L}} = {\downarrow_{TC_{\mathbb{L}}}} T\{1_{\mathbb{L}}\},$$

since $1_{\mathbb{L}} \in C_{\mathbb{L}}$ as $C_{\mathbb{L}}$ is a sublattice of $\mathbb{L}$. Since we assumed that T maps finite sets to finite sets, the set $T\{1_{\mathbb{L}}\}$ must be finite. It now follows from a straightforward generalization of (Vickers, 2006, Proposition 11) that $\mathrm{Fr}\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ is a compact frame. (The only change we need to make to (Vickers, 2006, Proposition 11) is to generalize from using single finite trees to using disjoint unions of $|T\{1_{\mathbb{L}}\}|$-many trees, so that one can cover each element of $T\{1_{\mathbb{L}}\}$.) □

We define $V_T^C\mathbb{L} := \mathrm{Fr}\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$, and for the time being we denote the insertion of generators by $\heartsuit\colon TC_{\mathbb{L}} \to V_T^C\mathbb{L}$. Our goal is now to show that $V_T\mathbb{L} \simeq V_T^C\mathbb{L}$. We will use a shortcut, exploiting the fact that both $V_T\mathbb{L}$ and $V_T^C\mathbb{L}$ have flat site presentations: we will define *suplattice* homomorphisms $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$ and $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$. We then show that $g' \circ f' = id$ and $f' \circ g' = id$, so that $V_T\mathbb{L}$ and $V_T^C\mathbb{L}$ are isomorphic as suplattices. It then follows from order theory that they are also isomorphic as frames. We start by defining a function $g\colon TC_{\mathbb{L}} \to V_T\mathbb{L}$, defined as

$$g\colon \alpha \mapsto \nabla\alpha.$$

Lemma 4.12. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor and let $\mathbb{L}$ be a frame. Then the function g defined above extends to a* suplattice *homomorphism $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$ such that $g' \circ \heartsuit = g$.*

$$\begin{tikzcd}
V_T^C\mathbb{L} \arrow[r, dotted, "g'"] & V_T\mathbb{L} \\
TC_{\mathbb{L}} \arrow[u, "\heartsuit"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. We need to show that $g\colon TC_{\mathbb{L}} \to V_T\mathbb{L}$ preserves the order on $TC_{\mathbb{L}}$ and preserves covers in to joins: if $\alpha \lhd_0^C \lambda^T(\Phi)$, where $\alpha \in TC_{\mathbb{L}}$, $\Phi \in TPC_{\mathbb{L}}$ and $\alpha \,\overline{T}{\leq}\, \bigvee(\Phi)$, then $g(\alpha) \leq \bigvee\{g(\beta) \mid \beta \in \lambda^T(\Phi)\}$. Both of these properties follow straightforwardly from the fact that $\langle TC_{\mathbb{L}}, \overline{T}{\leq}, \lhd_0^C\rangle$ is a substructure of $\langle TL, \overline{T}{\leq}, \lhd_0^{\mathbb{L}}\rangle$. $\square$

The next step is to define the suplattice homomorphism $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$. This requires a little more work than the definition of $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$, beginning with the following lemma.

Lemma 4.13. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor and let $\mathbb{L}$ be a compact frame. If $\alpha \in TC_{\mathbb{L}}$ and $\Phi \in TPC_{\mathbb{L}}$ such that $\alpha \,\overline{T}{\leq}\, T\bigvee(\Phi)$, then there exists $\Phi_\alpha \in TP_\omega C_{\mathbb{L}}$ such that $\Phi_\alpha \,\overline{T}{\subseteq_L}\, \Phi$ and $\alpha \,\overline{T}{\leq}\, T\bigvee(\Phi_\alpha)$.*

Proof. Since $\mathbb{L}$ is compact, we can show that

$$\text{for all } a \in C_{\mathbb{L}},\ a \text{ is compact.} \tag{51}$$

After all, if $a \in C_{\mathbb{L}}$ and $A \in PL$ such that $a \leq \bigvee A$, then also $1 \leq a \vee \neg a \leq \bigvee A \cup \{\neg a\}$, so by compactness of $\mathbb{L}$, there exists a finite $A' \subseteq A$ such that $a \vee \neg a \leq \bigvee A' \cup \{\neg a\}$. Consequently, $a \leq \bigvee A'$. Since A was arbitrary, it follows that a is compact.

We define

$$S := (\leq ; Gr(\textstyle\bigvee)^{\smile}) \restriction_{C_{\mathbb{L}} \times PC_{\mathbb{L}}};$$

so that $(a, A) \in S$ iff $a \in C_{\mathbb{L}}$, $A \in PC_{\mathbb{L}}$ and $a \leq \bigvee A$. By (51), we can define a function $h\colon S \to S$ where $h\colon (a, A) \mapsto (a', A')$ such that $a = a'$, $A' \subseteq A$, $a' \leq \bigvee A'$ (otherwise h would not be well-defined) and such that A' is finite, i.e. $A' \in P_\omega C_{\mathbb{L}}$. In other words, $h\colon S \to S$ is a function which assigns a finite subcover A' to a set of zero-dimensional opens A covering a zero-dimensional open element a . If we denote the projection functions of S as

$$C_{\mathbb{L}} \xleftarrow{p_1} S \xrightarrow{p_2} PC_{\mathbb{L}}$$

then we can encode the above-mentioned properties of h as follows:

$$\begin{aligned}
&\forall x \in S, p_1 \circ h(x) = p_1(x);\\
&\forall x \in S, p_2 \circ h(x) \subseteq p_2(x);\\
&\forall x \in S, p_2 \circ h(x) \in P_\omega C_{\mathbb{L}}.
\end{aligned}$$

By relation lifting, it follows that

$$\forall x \in TS, Tp_1 \circ Th(x) = Tp_1(x); \tag{52}$$

$$\forall x \in TS, Tp_2 \circ Th(x) \,\overline{T}{\subseteq}\, Tp_2(x); \tag{53}$$

$$\forall x \in TS, Tp_2 \circ Th(x) \in TP_\omega C_{\mathbb{L}}. \tag{54}$$

Finally, observe that it follows by relation lifting that

$$\forall \alpha \in TC_{\mathbb{L}}, \forall \Phi \in TPC_{\mathbb{L}},\ \alpha \,\overline{T}{\leq}\, \textstyle\bigvee(\Phi) \text{ iff } \alpha \,\overline{T}S\, \Phi.$$

Now take $\alpha \in TC_{\mathbb{L}}$ and $\Phi \in TPC_{\mathbb{L}}$ such that $\alpha \,\overline{T}{\leq}\, \bigvee(\Phi)$. Then by the above, we have $\alpha \,\overline{T}S\, \Phi$, so by definition of $\overline{T}$ there must exist some $x \in TS$ such that $Tp_1(x) = \alpha$ and $Tp_2(x) = \Phi$. We define

$\Phi_\alpha := Tp_2 \circ Th(x)$; observe that $Tp_1 \circ Th(x) = Tp_1(x) = \alpha$ by (52). Since Th is a function from TS to TS, we see that $\alpha\, \overline{T}S\, \Phi_\alpha$, so that $\alpha\, \overline{T}{\leq}\, T\bigvee(\Phi_\alpha)$. Moreover by (53) $\Phi_\alpha\, \overline{T}{\subseteq}\, \Phi$ and by (54), $\Phi_\alpha \in TP_\omega C_{\mathbb{L}}$. This concludes the proof. □

We now define a map $f\colon TL \to V_T^C\mathbb{L}$ by sending

$$f\colon \alpha \mapsto \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta\, \overline{T}{\leq}\, \alpha\}.$$

This will give us our suplattice homomorphism $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$.

Lemma 4.14. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving functor. If $\mathbb{L}$ is a compact zero-dimensional frame then $f\colon TL \to V_T^C\mathbb{L}$ defined above extends to a* suplattice *homomorphism $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$, where $f' \circ \nabla = f$.*

$$\begin{tikzcd} V_T\mathbb{L} \arrow[r, dotted, "f'"] & V_T^C\mathbb{L} \\ TL \arrow[u, "\nabla"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proof. In order to show that $f\colon TL \to V_T^C\mathbb{L}$ extends to a suplattice homomorphism, we need to show that f preserves the order on TL and f transforms covers into joins, i.e. that for all $(\alpha, \lambda^T(\Phi)) \in \lhd_0$, where $\alpha\, \overline{T}{\leq}\, T\bigvee(\Phi)$, we have $f(\alpha) \leq \bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\}$. To see why f is order-preserving, suppose that $\alpha_0, \alpha_1 \in TL$ and that $\alpha_0\, \overline{T}{\leq}\, \alpha_1$. Then

$$\begin{aligned} f(\alpha_0) &= \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta\, \overline{T}{\leq}\, \alpha_0\} && \text{by definition of } f, \\ &\leq \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta\, \overline{T}{\leq}\, \alpha_1\} && \text{since } \beta\, \overline{T}{\leq}\, \alpha_0\, \overline{T}{\leq}\, \alpha_1 \Rightarrow \beta\, \overline{T}{\leq}\, \alpha_1, \\ &= f(\alpha_1) && \text{by definition of } f. \end{aligned}$$

Before we go ahead and show that f transforms covers $\alpha \lhd_0 \lambda^T(\Phi)$ into joins, we show that the expression $\bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\}$ can be simplified:

$$\forall \Phi \in TPL,\, \bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\} = \bigvee\{\heartsuit\beta \mid \beta \in \lambda^T(T{\Downarrow}(\Phi))\}. \tag{55}$$

To see why, observe that

$$\begin{aligned} &\bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\} \\ &= \bigvee\left\{\bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \leq \gamma\} \mid \gamma \in \lambda^T(\Phi)\right\} && \text{by definition of } f, \\ &= \bigvee\left\{\bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \leq \gamma\} \mid \gamma\, \overline{T}{\in}\, \Phi\right\} && \text{by definition of } \lambda^T, \\ &= \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \exists\gamma\, \overline{T}{\in}\, \Phi,\, \beta \leq \gamma\} && \text{by associativity of } \textstyle\bigvee, \\ &= \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta\, \overline{T}{\leq}\,;\, \overline{T}{\in}\, \Phi\} && \text{by def. of relation composition}, \\ &= \bigvee\{\heartsuit\beta \mid \beta \in \lambda^T(T{\Downarrow}(\Phi))\} && \text{by Lemma 4.7(3).} \end{aligned}$$

Let $\alpha \in TL$ and $\Phi \in TPL$ such that $\alpha\, \overline{T}{\leq}\, T\bigvee(\Phi)$; we need to show that $f(\alpha) \leq \bigvee\{f(\gamma) \mid \gamma \in \lambda^T(\Phi)\}$. By (55) it suffices to show that

$$f(\alpha) \leq \bigvee\{\heartsuit\gamma \mid \gamma \in \lambda^T(T{\Downarrow}(\Phi))\}. \tag{56}$$

Recall that $f(\alpha) = \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\, \beta \leq \alpha\}$. We will show that

$$\forall \beta \in TC_{\mathbb{L}},\, \beta\, \overline{T}{\leq}\, \alpha \Rightarrow \heartsuit\beta \leq \bigvee\{\heartsuit\gamma \mid \gamma \in \lambda^T(T{\Downarrow}(\Phi))\}. \tag{57}$$

Suppose that $\beta \in TC_{\mathbb{L}}$ and that $\beta\, \overline{T}{\leq}\, \alpha$. Then since we assumed that $\alpha\, \overline{T}{\leq}\, T\bigvee(\Phi)$, it follows that $\beta\, \overline{T}{\leq}\, T\bigvee(\Phi)$. By Lemma 4.7(2), we know that $T\bigvee(\Phi) = T\bigvee \circ T{\Downarrow}(\Phi)$, so we see that

$$\beta\, \overline{T}{\leq}\, T\bigvee \circ T{\Downarrow}(\Phi).$$

Now since $T{\Downarrow}(\Phi) \in TPC_{\mathbb{L}}$, we can now apply Lemma 4.13 to conclude that there must be some $\Phi' \in TP_\omega C_{\mathbb{L}}$ such that $\Phi' \,\overline{T}{\subseteq}\, T{\Downarrow}(\Phi)$ and $\beta \,\overline{T}{\leq}\, \bigvee\Phi'$. Now it follows by definition of $\lhd_0^C$ that $\beta \lhd_0^C \lambda^T(\Phi')$. Now

$$\begin{aligned} \heartsuit\beta &\leq \bigvee\{\heartsuit\gamma \mid \gamma \in \lambda^T(\Phi')\} && \text{since } \beta \lhd_0^C \lambda^T(\Phi'), \\ &\leq \bigvee\{\heartsuit\gamma \mid \gamma \in \lambda^T(T{\Downarrow}(\Phi))\} && \text{by L. 3.24 since } \Phi' \,\overline{T}{\subseteq}\, T{\Downarrow}(\Phi). \end{aligned}$$

Since $\beta \in TC_{\mathbb{L}}$ was arbitrary it follows that (57) holds; consequently, (56) holds so that we may indeed conclude that f transforms covers into joins. We conclude that $f\colon TL \to V_T^C\mathbb{L}$ extends to a suplattice homomorphism $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$. ☐

Now that we have established the existence of suplattice homomorphisms $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$ and $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$, we are ready to prove the theorem of this subsection.

Theorem 4.15. *Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a standard, finitary, weak pullback-preserving set functor which maps finite sets to finite sets and let $\mathbb{L}$ be a frame. If $\mathbb{L}$ is compact and zero-dimensional then so is $V_T\mathbb{L}$.*

Proof. It follows by Theorem 4.9 that $V_T\mathbb{L}$ is zero-dimensional. To show that $V_T\mathbb{L}$ is compact, it suffices to show that $V_T\mathbb{L} \simeq V_T^C\mathbb{L}$ by Lemma 4.11. We will establish that $V_T\mathbb{L} \simeq V_T^C\mathbb{L}$ by showing that $g'\colon V_T^C\mathbb{L} \to V_T\mathbb{L}$ and $f'\colon V_T\mathbb{L} \to V_T^C\mathbb{L}$ are *suplattice* isomorphisms, because $g' \circ f' = id_{V_T\mathbb{L}}$ and $f' \circ g' = id_{V_T^C\mathbb{L}}$. This is sufficient since by order theory, any suplattice isomorphism is also a frame isomorphism. We begin by making the following claim:

$$\forall \alpha \in TL,\ g' \circ f(\alpha) = \nabla\alpha. \tag{58}$$

After all, if $\alpha \in TL$ then

$$\begin{aligned} g' \circ f(\alpha) &= g'\left(\bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\ \beta \,\overline{T}{\leq}\, \alpha\}\right) && \text{by definition of } f, \\ &= \bigvee\{g'(\heartsuit\beta) \mid \beta \in TC_{\mathbb{L}},\ \beta \,\overline{T}{\leq}\, \alpha\} && \text{since } g' \text{ preserves } \textstyle\bigvee, \\ &= \bigvee\{g(\beta) \mid \beta \in TC_{\mathbb{L}},\ \beta \,\overline{T}{\leq}\, \alpha\} && \text{by Lemma 4.12}, \\ &= \bigvee\{\nabla\beta \mid \beta \in TC_{\mathbb{L}},\ \beta \,\overline{T}{\leq}\, \alpha\} && \text{by definition of } g, \\ &= \nabla\alpha && \text{by Lemma 4.7(1)}. \end{aligned}$$

It follows that (58) holds. Conversely, we claim that

$$\forall \alpha \in TC_{\mathbb{L}},\ f' \circ g(\alpha) = \heartsuit\alpha. \tag{59}$$

This is also not hard to see. Take $\alpha \in TC_{\mathbb{L}}$, then

$$\begin{aligned} f' \circ g(\alpha) &= f'(\nabla\alpha) && \text{by definition of } g, \\ &= f(\alpha) && \text{by Lemma 4.14}, \\ &= \bigvee\{\heartsuit\beta \mid \beta \in TC_{\mathbb{L}},\ \beta \,\overline{T}{\leq}\, \alpha\} && \text{by definition of } f, \\ &= \heartsuit\alpha && \text{since } \alpha \in TC_{\mathbb{L}} \text{ and } \heartsuit \text{ is order-preserving}. \end{aligned}$$

It follows that (59) holds. Now we see that for all $\alpha \in TL$,

$$\begin{aligned} g' \circ f'(\nabla\alpha) &= g' \circ f(\alpha) && \text{since } f' \circ \nabla = f, \\ &= \nabla\alpha && \text{by (58)}, \\ &= id_{V_T\mathbb{L}}(\nabla\alpha). \end{aligned}$$

In other words, we see that $g' \circ f'$ and $id_{V_T\mathbb{L}}$ agree on the generators of $V_T\mathbb{L}$; it follows that $g' \circ f' = id_{V_T\mathbb{L}}$. An analogous argument shows that $f' \circ g' = id_{V_T^C\mathbb{L}}$. We conclude that $V_T\mathbb{L}$ and $V_T^C\mathbb{L}$ are isomorphic as suplattices and consequently also as frames; it follows that $V_T\mathbb{L}$ is compact. ☐

5. Future work

To finish off the paper, we list some open problems and directions for future work.

Preservation properties. The main technical problems that we would like to solve concern further preservation properties of our construction. In particular, we are very eager to find out for which functors T the T-power construction preserves compactness, or the combination of compactness and regularity. Observe that any functor satisfying this property must map finite sets to finite sets; if TA would be infinite for some finite A subset of $\mathbb{L}$, then we may have $1_{V_T\mathbb{L}} = \bigvee\{\nabla\alpha \mid \alpha \in A\}$, without there being a finite subcover. We conjecture that this condition (that is, of T restricting to finite sets) is in fact not only necessary, but also sufficient to prove the preservation of compactness.

Functorial properties. In section 3.4 we saw that certain natural transformations $\rho\colon T' \to T$ induce natural transformations $\widehat{\rho}\colon V_T \to V_{T'}$, with the unit of the Vietoris comonad V_{P_ω} providing an instance of this phenomenon. There are some natural open questions related to this. In particular, we are interested whether, in the case that T is actually a *monad*, it holds that V_T is a co-monad.

Another question related to the natural transformation $\widehat{\rho}$ is whether $\widehat{\rho}_{\mathbb{L}}\colon V_T\mathbb{L} \to V_{T'}\mathbb{L}$ always has a right adjoint, see Remark 3.22.

Spatiality and compact Hausdorff spaces. Palmigiano & Venema (Palmigiano & Venema, 2007) introduce a lifting construction on Chu spaces to prove that for Stone spaces, the Vietoris construction can be generalized from the power set case to an arbitrary set functor T (meeting the same constraints as in the current paper). Can we generalize to arbitrary topological spaces, or at least to compact Hausdorff spaces?

Assume that, for any functor T mapping finite sets to finite sets, we can prove that our T-powerlocale construction V_T preserves the combination of compactness and regularity. Then, using the well-known duality between compact regular locales and compact Hausdorff spaces, we obtain a Vietoris-like functor on compact Hausdorff spaces for free. The question is then whether we can give a more direct, insightful description of this functor.

Locales and constructivity. In this paper, we have mostly adopted a frame- rather than a locale-oriented perspective. Theorem 3.21 suggests however, that if one wants to understand the relationship between coalgebra functors $T\colon \mathsf{Set} \to \mathsf{Set}$ and the V_T construction, one should think of V_T as a functor on *locales*, since natural transformations $T' \to T$ satisfying the conditions of Th. 3.21 correspond to frame natural transformations $V_T \to V_{T'}$. It would be interesting to pursue this idea further, especially in conjunction with the use of *constructive mathematics*. We have seen that certain constructive techniques, such as frame, flat site and preframe presentations, can be brought over to the framework of coalgebraic logic. Making the entire approach of this paper constructive would be a lot of work; we believe however that this would be a promising line of further research.

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