

Canonical extension and canonicity via DCPO presentations

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Abstract

The canonical extension of a lattice is in an essential way a two-sided completion. Domain theory, on the contrary, is primarily concerned with one-sided completeness. In this paper, we show two things. Firstly, that the canonical extension of a lattice can be given an asymmetric description in two stages: a free co-directed meet completion, followed by a completion by *selected* directed joins. Secondly, we show that the general techniques for dcpo presentations of dcpo algebras used in the second stage of the construction immediately give us the well-known canonicity result for bounded lattices with operators.

Keywords: dcpo presentation, dcpo algebra, lattice theory, canonical extension, canonicity

1. Introduction

Domain theory on the one side and canonical extensions and canonicity on the other side are topics that have played a fundamental role in non-classical logic and its computer science applications for a long time. Domain theory has been intrinsically tied to foundational issues in computer science since it was introduced by Dana Scott in the late 1960s in order to provide semantics for the lambda calculus [17]. The solution of domain equations and the modern techniques for dcpo presentations are particularly important tools [1, 14]. Canonical

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extensions in their algebraic form were first introduced by Jónsson and Tarski in 1951 with the hopes of giving a representation theorem for relation algebras [13]. However, they were later realised to be closely related to the very important canonical model construction in logic and thus to issues concerning relational semantics for a plethora of logics important in computer science applications such as modal logics [12]. The algebraic approach to canonical extensions and questions of canonicity have been revitalised over the last few decades after the theory was extended beyond the setting of Boolean-based logics and additional operations that preserve joins in each coordinate. The initial step in this development was the realisation that Scott continuity plays a central role in the theory [7]. Apart from this one fundamental connection, the two topics have not had much to do with each other and any more tangible connections have remained hidden. This is somewhat remarkable in light of the central role Stone duality plays in both domain theory [2] *and* canonical extension. We will briefly touch upon the interaction between Stone duality, domain theory and canonical extensions in Section 1.2 below.

On a more directly mathematical level, there are also other reasons to seek to understand the connections between domain theory and the theory of canonical extensions. Completing, or directedly completing, posets may be done freely if we only consider one-sided limits in the form either of joins or meets and this is fundamental to the theory of domains and the related theory of frames as studied in pointfree topology. However, unrestricted two-sided free completions do not exist. Canonical extensions may be viewed as the second level (after MacNeille completion) of two-sided completions obtained by restricting the alternations of joins and meets required to generate the completion [10]. As such, they are certainly dcpos, and in the distributive setting, algebraic domains and they remain so when turned upside down. This begs the question of understanding these two-sided completions relative to the one-sided completion techniques that are so central in domain theory. In this spirit, this paper is an answer to a question raised by Achim Jung during his talk at TACL2009 of the relation between his results with Moshier and Vickers in [14] and canonical extensions. To be specific, we show that the canonical extension of a lattice can be given an asymmetric description in two stages: a free co-directed meet completion followed by a completion by selected directed joins as made possible by the methods of dcpo presentations. In addition, we show that the pivotal 1994 canonicity result [7] that introduced Scott continuity into the theory of canonical extensions may in fact be seen as a special case of the theorem on representations of dcpo algebras given in [14] thus making the connection between the two fields quite explicit. In obtaining the 1994 canonicity result from the one-sided theory, the setting of dcpo algebras rather than just suplattice algebras is crucial as the former is needed in order to have a result on the lifting of operations available (see Remark 1 in Section 3 below).

The organization of this paper is as follows: first, we provide brief discussions about the background of canonical extension, both in relation to Stone duality and in relation to logic. After that, in Section 2, we provide preliminaries on dcpo presentations, dcpo algebras, free directed completions and canonical

extensions. The main results are presented in Section 3; after which we conclude the article with a discussion in Section 4.

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1.1. Canonical extension and Stone duality

At its base, canonical extension is an algebraic way of talking about Stone’s duality for bounded distributive lattices. To see this consider the following square of functors for which both the inner and the outer square commute

$$\begin{array}{ccc}
 \text{DL} & \begin{array}{c} \xrightarrow{S} \\ \xleftarrow{CO} \end{array} & \text{Stone} \\
 \begin{array}{c} \uparrow \sigma \\ \downarrow \end{array} & & \begin{array}{c} \uparrow \beta \\ \downarrow \end{array} \\
 \text{DL}^+ & \begin{array}{c} \xrightarrow{J^\infty} \\ \xleftarrow{U} \end{array} & \text{Pos}
 \end{array}$$

Here the upper pair of functors gives the Stone duality for bounded distributive lattices and spectral spaces, and the lower pair of functors gives the ‘discrete’ duality between completely distributive algebraic lattices (or complete lattices join-generated by their completely join-prime elements) and partially ordered sets. This second duality generalises the very well-known duality between complete and atomic Boolean algebras and Sets. On objects, it sends a completely distributive algebraic lattice (DL^+) to its poset of completely join-irreducible elements and a poset to its lattice of upsets.

In the vertical direction, we have natural forgetful functors: DL^+ s are in particular DLs, and topological spaces give rise to posets via the specialisation order: $x \leq y$ if and only if every open containing x also contains y . These forgetful functors go in opposite directions so they are obviously not translations of each other across the dualities. Instead, they translate to left adjoints of each other across the dualities. This brings us to the canonical extension. The forgetful functor $\text{DL}^+ \rightarrow \text{DL}$ that embeds DL^+ as a non-full subcategory of DL has a left adjoint $\sigma : \text{DL} \rightarrow \text{DL}^+$ and its dual incarnation is the forgetful functor from Stone spaces to posets. Moreover, this left adjoint $\sigma : \text{DL} \rightarrow \text{DL}^+$ is a reflector. Thus we have, for each DL, an embedding $A \hookrightarrow A^\sigma$; this embedding is the *canonical extension*. The dual incarnation of the inclusion from DL^+ to DL is the left adjoint of the forgetful functor from the category of Stone (=spectral) spaces to the category of posets. In the distributive lattice setting this left adjoint was first identified by Banaschewski in [3] and in the Boolean setting it is the very well-known Stone-Čech compactification.

We reiterate that both of the inclusions, $\text{DL}^+ \rightarrow \text{DL}$ and the one of spectral spaces in posets, are inclusions as *non-full subcategories*: a DL^+ morphism is not just a bounded lattice homomorphism but a complete lattice homomorphism; similarly there are maps between spectral spaces which preserve the specialisation order without being continuous. As a consequence, even for objects in

the subcategories on either side of the square, the reflectors need not be the identity. To wit, for an infinite powerset Boolean algebra, B , the canonical extension will be the powerset of the set of all ultrafilters of B – a significantly larger Boolean algebra. Dually, this corresponds to the fact that for an infinite Boolean space, the Stone-Čech compactification of the underlying set, viewed as a discrete space, will be much larger than the original space.

Returning to our square of functors, note that the commutativity of the square means that we can understand A^σ in terms of the dual space $S(A) = (X, \tau)$. That is, $A^\sigma = \mathcal{U}(X, \leq)$ is the lattice of upsets of the dual space of A equipped with the specialisation order of the Stone topology τ . The embedding of A in its canonical extension in this description is given by the Stone embedding map $a \mapsto \hat{a}$ which maps each element of the lattice to the corresponding compact open upset. So canonical extension can be obtained via duality and for this reason it is often referred to as the ‘double dual’ in the logic literature.

Most interestingly, the converse is also true: It is possible to reconstruct the dual space of A from the canonical extension $A \hookrightarrow A^\sigma$ and this is why we can claim that the theory of canonical extensions may be seen as an algebraic formulation of Stone/Priestley duality. Given the canonical extension $A \hookrightarrow A^\sigma$ of a DL, we obtain the dual space of A by applying the discrete duality to obtain the set $X = J^\infty(A^\sigma)$. The topology is then generated by the ‘shadows’ of the elements of A on X , that is, by the sets $\hat{a} = \{x \in X \mid x \leq a\}$ where a ranges over A .

We point out two advantages of the canonical extension approach to duality. Firstly, canonical extension is particularly well-suited for studying additional operations on lattices or Boolean algebras. This was the original purpose for canonical extensions and their scope has been expanded in a modular fashion [8, 6, 5] in order to provide representation theorems for lattice- and even poset-based algebras. The two-sided aspect is particularly important when additional operations that are order-reversing are present. Secondly, although the classical existence proof [13] for the canonical extension uses the Prime Filter Theorem, it is now known [6] that one can develop the theory of canonical extensions without invoking the Axiom of Choice.

1.2. Canonical extension and logic

In logic and computer science, Stone duality is central in many ways. A landmark paper in setting this out in the clearest of terms is Abramsky’s paper [2] where he shows how Stone duality for distributive lattices allows us to connect specification languages with denotational semantics. The role of Stone duality is similar in modal logic in the sense that it connects specification and state-based models, but the two approaches differ in the way they manage to factor out the topology inherent in Stone duality. In domain theory, one restricts to very special lattices and spaces for which the topology is determined by the specialisation order. In modal logic, one focuses on logics for which the topology ‘factors out’ in the sense that forgetting it does not change the logic.

Canonical extensions are particularly pertinent for several reasons. One is that we usually have additional operations, like modalities, negations, or

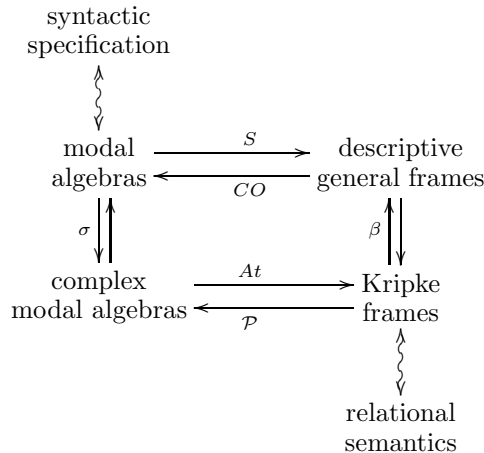
implications and the translation of such structure as well as their equational properties to the dual side is more easily understood by going via canonical extension and correspondence across the discrete duality [9]. A second and very important reason that canonical extensions play a central role in the study of various logics is that they are centrally related to relational semantics for these logics.

We illustrate this with the example of classical propositional modal logic, and we will give a very brief impression of the role that canonical extensions play in the model theory of modal logic as it is described in [4]. We will consider the following two natural semantics for modal logic:

- Kripke frames, which are set-based transition systems or coalgebras for the covariant powerset functor to be more precise,
- modal algebras: Boolean algebras with an additional unary (finite join preserving) operation, meant to interpret the modal diamond operator.

The former provide the natural semantics for modal logic and are central in various state-based models in computer science. The latter provide a specification language for these systems and often correspond to the syntactic description of the pertinent logics.

Thus, for classical modal logics, the restriction of the above square to Boolean algebras is the appropriate one, and then the additional structure is superposed: a modal operator on the Boolean algebras translates to a binary relation with certain topological properties on the dual spaces - this is what is known as descriptive general frames. Forgetting the topology yields Kripke frames, which are in a discrete duality with complex modal algebras. Note that while the inner and outer square still commute the vertical functors are only reflectors for the underlying Boolean algebras: this is extended Stone duality and not natural duality for modal algebras.



The central importance of canonical extension in this setting comes from the fact, mentioned above, that the two important spots in the above diagram

are the upper left and the lower right: the upper left corresponds to the syntactic specification of the logic; the lower right to the semantic specification. Thus moving horizontally is not enough; we must also move up and down. In addition, we claim that the route down-and-over may be viewed as separating the issues involved better than the route over-and-down. To this end, one can think of the upper left-hand corner as the finitary description of the base of a topological space, and of the lower right-hand corner as the points underlying the space. Taking the canonical extension, i.e. going down from the upper left hand corner, corresponds to augmenting the finitary description of the base with infinitary (but point-free) information; subsequently going over adds points to the picture. If we go over and down, already the first step (of going over) simultaneously moves us to a topological and point-based perspective, while going down just forgets part of what we have worked hard to identify in the topological duality. Note that this separation of topological and contravariant content of the topological duality is even useful if our final goal is full-fledged topological duality (i.e., the upper right-hand corner) and not just the lower right-hand corner where the topology has been removed since, as we outlined in the previous subsection, the canonical extension, $A \hookrightarrow A^\sigma$ (but not A^σ alone) contains all the topological information of the topological duality in a point-free and co-variant way.

Finally, consider the question of logical completeness. Given the way Kripke semantics is defined, a formula ϕ is valid in a structure if and only if the identity $\phi \approx 1$ holds in the corresponding complex algebra. This is essentially the definition. On the other hand, a syntactic specification of a modal logic is typically an equational theory, Σ , of modal algebras. Thus soundness with respect to a class \mathcal{K} of structures means that the complex algebras of the structures in \mathcal{K} all are models of Σ . Completeness, in the contrapositive, means that an equation that is not a consequence of Σ is violated in the complex algebra of some $K \in \mathcal{K}$. Canonicity of Σ means that the class of models of Σ is closed under canonical extension. Any equation that isn't a consequence of a theory Σ is violated by some abstract algebra model of Σ and thus also by its canonical extension. If Σ is canonical then this canonical extension is a model of the theory in which the given equation is violated. In this way canonicity implies that the logic possesses complete Kripke semantics. One should note that not all modal logics are canonical but most of the standard ones are. However, even in the absence of canonicity, it is clear that canonical extensions are pertinent since they provide an account of the connection between the upper left and lower right corner of the diagram.

2. Preliminaries

We collect here the main facts on dcpo completions, free co-directed completions, and canonical extensions that we will need and give specific references to where one can find proofs.

2.1. DCPO and suplattice presentations

The following facts about dcpo presentations, suplattice presentations, and dcpo algebras may be found in [14].

Definition 1. A *dcpo presentation* is a triple $\langle P; \sqsubseteq, C \rangle$ where

- $\langle P, \sqsubseteq \rangle$ is a preorder,
- $C \subseteq P \times \mathcal{P}(P)$ is a family of *covers*, where U is directed for every $(x, U) \in C$. We write $x \triangleleft U$ if $(x, U) \in C$.

Let $\langle D, \leq \rangle$ be a dcpo and let $f: P \rightarrow D$ be an order-preserving map. We say f *preserves covers* if for all $x \triangleleft U$ it is true that $f(x) \leq \bigvee_{y \in U} f(y)$. Note that, from here on, we will refer to maps preserving either an order or a preorder as order-preserving in order to lighten the notation.

A *suplattice* is a complete join-semilattice; the appropriate homomorphisms between suplattices are those maps which preserve all joins. If we replace ‘dcpo’ by ‘suplattice’ in Definition 1 and if we drop the assumption that each U above is directed, we obtain the definition for a *suplattice presentation*. Observe that every dcpo presentation is also a suplattice presentation.

Definition 2. A dcpo \overline{P} is *freely generated* by the dcpo presentation $\langle P; \sqsubseteq, C \rangle$ if there is a map $\eta: P \rightarrow \overline{P}$ that preserves covers, and for every dcpo $\langle D, \leq \rangle$ and cover-preserving map $f: P \rightarrow D$ there is a unique Scott-continuous map $\overline{f}: \overline{P} \rightarrow D$ such that $\overline{f} \circ \eta = f$.

Again, if we replace ‘dcpo’ with ‘suplattice’ and ‘Scott-continuous map’ by ‘suplattice homomorphism’ above, we obtain the definition of a suplattice freely generated by a suplattice presentation. We will now describe how freely generated dcpos and suplattices are obtained in [14].

Definition 3. A *C-ideal* of P is a set $X \subseteq P$ which is downward closed and closed under covers, i.e. for all $x \triangleleft U$, if $U \subseteq X$ then $x \in X$. We denote the set of all C -ideals of P by $C\text{-Idl}(P)$.

An arbitrary intersection of C -ideals is again a C -ideal; thus the collection of all C -ideals of $\langle P; \sqsubseteq, C \rangle$ forms a complete lattice $C\text{-Idl}(P)$ and we can denote by $\langle X \rangle$ the smallest C -ideal containing X for any $X \subseteq P$; we will abbreviate $\langle \{x\} \rangle$ as $\langle x \rangle$. Observe that $\downarrow X \subseteq \langle X \rangle$. We will denote meets and joins in $C\text{-Idl}(P)$ by \bigwedge and \bigvee , respectively. Note that for all $S \subseteq C\text{-Idl}(P)$, $\bigwedge S = \bigcap S$ and $\bigvee S = \langle \bigcup S \rangle$.

Proposition 1 ([14], Proposition 2.5). *Let $\langle P; \sqsubseteq, C \rangle$ be a suplattice presentation. Then $\langle C\text{-Idl}(P), \sqsubseteq \rangle$ is the suplattice freely generated by $\langle P; \sqsubseteq, C \rangle$, where $\eta: P \rightarrow \overline{P}$ is defined by $\eta: x \mapsto \langle x \rangle$.*

Definition 4. Given a dcpo presentation $\langle P; \sqsubseteq, C \rangle$, we define

$$\overline{P} = \bigcap \{ X \subseteq C\text{-Idl}(P) \mid X \text{ is closed under directed joins and } \langle x \rangle \in X \text{ for all } x \in P \}.$$

Proposition 2 ([14], Theorem 2.7). *Let $\langle P; \sqsubseteq, C \rangle$ be a dcpo presentation. Then $\langle \overline{P}, \sqsubseteq \rangle$ is the dcpo freely generated by $\langle P; \sqsubseteq, C \rangle$, where $\eta: P \rightarrow \overline{P}$ is defined by $\eta: x \mapsto \langle x \rangle$.*

Observe that it is ‘hard’ to tell which C -ideals belong to \overline{P} ; see the comments at the end of Section 2 of [14].

2.2. DCPO algebras

We now turn to algebras. A *pre-ordered algebra* for a set of operation symbols Ω with arities $\alpha: \Omega \rightarrow \mathbb{N}$ consists of a pre-order $\langle P, \sqsubseteq \rangle$ and order-preserving maps $\omega_P: P^{\alpha(\omega)} \rightarrow P$ for $\omega \in \Omega$. For dcpo presentations $\langle P_1; \sqsubseteq, C_1 \rangle, \dots, \langle P_n; \sqsubseteq, C_n \rangle, \langle P'; \sqsubseteq, C' \rangle$ we write $x_i \triangleleft_i U_i$ if $(x_i, U_i) \in C_i$. An order-preserving map $f: P_1 \times \dots \times P_n \rightarrow P'$ is called *cover-stable* if for all $1 \leq i \leq n$, all $(x_1, \dots, x_n) \in P_1 \times \dots \times P_n$ and all $U_i \sqsubseteq P_i$ such that $x_i \triangleleft U_i$, we have

$$f(x_1, \dots, x_n) \triangleleft' \{f(x_1, \dots, x_{i-1}, y, x_{i+1}, x_n) \mid y \in U_i\}.$$

Proposition 3 ([14], Theorem 3.6). *If $f: P_1 \times \dots \times P_n \rightarrow P'$ is cover-stable and order-preserving, then the function $\overline{f}: \overline{P}_1 \times \dots \times \overline{P}_n \rightarrow \overline{P}'$, defined by*

$$\overline{f}: (X_1, \dots, X_n) \mapsto \langle \{f(x_1, \dots, x_n) \mid (x_1, \dots, x_n) \in X_1 \times \dots \times X_n\} \rangle,$$

is a well-defined and Scott-continuous extension of f (and is unique as such).

Proposition 4 ([14], Proposition 4.2). *Consider a structure $\langle P; \sqsubseteq, C, (\omega_P)_{\omega \in \Omega} \rangle$ such that $\langle P; \sqsubseteq, C \rangle$ is dcpo presentation and $\langle P; \sqsubseteq, (\omega_P)_{\omega \in \Omega} \rangle$ is a preordered algebra. Let $s(x_1, \dots, x_n)$ and $t(x_1, \dots, x_n)$ be n -ary Ω -terms. If for every $\omega \in \Omega$, $\omega_P: P^{\alpha(\omega)} \rightarrow P$ is cover-stable, then we can define an Ω -algebra structure on \overline{P} by taking $\omega_{\overline{P}} := \overline{\omega_P}$ and $P \models s \preceq t$ implies $\overline{P} \models s \preceq t$.*

2.3. Free directed completions

The free directed join completion and the free co-directed meet completion of a poset are given by the posets of filters and of ideals of the poset, respectively. For our purposes, an abstract characterisation of these completions will be important. The following results date back to [16] and are very well known. Sources for this material are [15], Section 6, and [11], Sections I-4 and IV-1 and [10].

Definition 5. Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset. By $\uparrow_{\mathbb{P}}: \mathbb{P} \rightarrow \mathcal{F}(\mathbb{P})$ we denote the *co-directed meet completion* of \mathbb{P} , which is characterized by the following properties:

1. $\langle \mathcal{F}(\mathbb{P}), \leq \rangle$ is a co-dcpo,
2. $\uparrow_{\mathbb{P}}: \mathbb{P} \rightarrow \mathcal{F}(\mathbb{P})$ is an order-embedding,
3. for every $x \in \mathcal{F}(\mathbb{P})$, $\{a \in P \mid x \leq \uparrow_{\mathbb{P}} a\}$ is co-directed and $x = \bigwedge \{\uparrow_{\mathbb{P}} a \mid x \leq \uparrow_{\mathbb{P}} a\}$,
4. for all co-directed $S \subseteq \mathcal{F}(\mathbb{P})$ and all $a \in P$, if $\bigwedge S \leq \uparrow_{\mathbb{P}} a$ then there exists $s \in S$ such that $s \leq \uparrow_{\mathbb{P}} a$.

Proposition 5. *If \mathbb{P} and \mathbb{Q} are posets, then $\mathcal{F}(\mathbb{P} \times \mathbb{Q}) \cong \mathcal{F}(\mathbb{P}) \times \mathcal{F}(\mathbb{Q})$.*

If $f: \mathbb{P} \rightarrow \mathbb{Q}$ is an order-preserving map between posets, then f has a unique co-Scott continuous extension, $f^{\mathcal{F}}: \mathcal{F}(\mathbb{P}) \rightarrow \mathcal{F}(\mathbb{Q})$, defined as follows:

$$f^{\mathcal{F}}: x \mapsto \bigwedge \{\uparrow_{\mathbb{Q}} f(a) \mid x \leq \uparrow_{\mathbb{P}} a\}.$$

Given an ordered algebra $\mathbb{A} = \langle A, \leq; (\omega_{\mathbb{A}})_{\omega \in \Omega} \rangle$ such that every $\omega_{\mathbb{A}}$ is order-preserving, we can define an algebra structure on $\mathcal{F}(\mathbb{A})$ by taking $\omega_{\mathcal{F}(\mathbb{A})} := (\omega_{\mathbb{A}})^{\mathcal{F}}$.

Proposition 6. *Let $s(x_1, \dots, x_n)$ and $t(x_1, \dots, x_n)$ be n -ary Ω -terms and let \mathbb{A} be an ordered Ω -algebra. If $\mathbb{A} \models s \preceq t$ then also $\mathcal{F}(\mathbb{A}) \models s \preceq t$.*

Proposition 7. *Let $\mathbb{A} = \langle A; \wedge, \vee, 0, 1 \rangle$ be a lattice. Then $\langle \mathcal{F}(\mathbb{A}), \wedge^{\mathcal{F}}, \vee^{\mathcal{F}}, 0, 1 \rangle$ is a (complete) lattice and $\uparrow_{\mathbb{A}}: \mathbb{A} \rightarrow \mathcal{F}(\mathbb{A})$ is a lattice embedding.*

We denote the meet and join operation of $\mathcal{F}(\mathbb{A})$ by \wedge and \vee respectively; also, we will let \bigwedge denote arbitrary meets in $\mathcal{F}(\mathbb{A})$. Given lattices $\mathbb{A}_1, \dots, \mathbb{A}_n, \mathbb{B}$, we say $f: A_1 \times \dots \times A_n \rightarrow B$ is an operator if for every $1 \leq i \leq n$, all $a_i, b_i \in A_i$ and all $a_j \in A_j, j \neq i$, we have

$$f(a_1, \dots, a_{i-1}, a_i \vee b_i, a_{i+1}, \dots, a_n) = f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \vee f(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n).$$

Proposition 8. *If $f: \mathbb{A}_1 \times \dots \times \mathbb{A}_n \rightarrow \mathbb{B}$ is an operator, then so is $f^{\mathcal{F}}: \mathcal{F}(\mathbb{A}_1) \times \dots \times \mathcal{F}(\mathbb{A}_n) \rightarrow \mathcal{F}(\mathbb{B})$.*

2.4. Canonical extension

Below we introduce the canonical extension of a lattice and the canonical extension of an order-preserving map between lattices [6]. Let \mathbb{A} be a lattice. A lattice completion of \mathbb{A} is a lattice embedding $e: \mathbb{A} \rightarrow \mathbb{C}$ of \mathbb{A} into a complete lattice \mathbb{C} . Two completions of \mathbb{A} , $e_1: \mathbb{A} \rightarrow \mathbb{C}_1$ and $e_2: \mathbb{A} \rightarrow \mathbb{C}_2$, are isomorphic if there exists a lattice isomorphism $f: \mathbb{C}_1 \rightarrow \mathbb{C}_2$ such that $f e_1 = e_2$.

Definition 6. Let $e: \mathbb{A} \rightarrow \mathbb{C}$ be a lattice completion of \mathbb{A} . We call $e: \mathbb{A} \rightarrow \mathbb{C}$ a canonical extension of \mathbb{A} if the following two conditions hold:

- (density) for all $u, v \in \mathbb{C}$ such that $u \not\leq v$, there exist a filter $F \subseteq A$ and an ideal $I \subseteq A$ such that

$$\bigwedge e[F] \leq u, \bigwedge e[F] \not\leq v, v \leq \bigvee e[I] \text{ and } u \not\leq \bigvee e[I];$$

- (compactness) for all ideals $I \subseteq A$ and all filters $F \subseteq A$, if $\bigwedge e[F] \leq \bigvee e[I]$ then there exist $b \in F$ and $a \in I$ such that $b \leq a$.

Proposition 9 ([6], Propositions 2.6 and 2.7). *Every lattice \mathbb{A} has a canonical extension, denoted $e_{\mathbb{A}}: \mathbb{A} \rightarrow \mathbb{A}^{\sigma}$. Moreover, $e_{\mathbb{A}}: \mathbb{A} \rightarrow \mathbb{A}^{\sigma}$ is unique up to isomorphism of completions.*

We will omit the subscript on $e_{\mathbb{A}}$ if it is clear from the context what \mathbb{A} is. Given $e: \mathbb{A} \rightarrow \mathbb{A}^\sigma$, we define $K(\mathbb{A}^\sigma) := \{\bigwedge e[F] \mid F \subseteq A \text{ a filter}\}$ to be the *closed elements* of \mathbb{A}^σ .

Definition 7. Let $f: \mathbb{A}_1 \times \cdots \times \mathbb{A}_n \rightarrow \mathbb{B}$ be an order-preserving map where $\mathbb{A}_1, \dots, \mathbb{A}_n$ and \mathbb{B} are lattices. We define $f^\sigma: \mathbb{A}_1^\sigma \times \cdots \times \mathbb{A}_n^\sigma \rightarrow \mathbb{B}^\sigma$ by first putting

$$f^\sigma: (x_1, \dots, x_n) \mapsto \bigwedge \{e_{\mathbb{B}}(f(a_1, \dots, a_n)) \mid (x_1, \dots, x_n) \leq (a_1, \dots, a_n)\}$$

for all tuples of closed elements $(x_1, \dots, x_n) \in K(\mathbb{A}_1^\sigma) \times \cdots \times K(\mathbb{A}_n^\sigma)$. We then define f^σ as follows on arbitrary tuples $(u_1, \dots, u_n) \in \mathbb{A}_1^\sigma \times \cdots \times \mathbb{A}_n^\sigma$:

$$f^\sigma: (u_1, \dots, u_n) \mapsto \bigvee \{f^\sigma(x_1, \dots, x_n) \mid (u_1, \dots, u_n) \geq (x_1, \dots, x_n) \in K(\mathbb{A}_1^\sigma) \times \cdots \times K(\mathbb{A}_n^\sigma)\}.$$

For information on the naturality of this definition in the distributive setting, see [8], Theorem 2.15.

3. A dcpo presentation of the canonical extension

Definition 8. Given a lattice \mathbb{A} , we define a dcpo presentation

$$\Delta(\mathbb{A}) := \langle \mathcal{F}(\mathbb{A}); \leq, C_{\mathbb{A}} \rangle$$

where

$$C_{\mathbb{A}} := \{(x, U) \in \mathcal{F}(\mathbb{A}) \times \mathcal{P}(\mathcal{F}(\mathbb{A})) \mid U \text{ non-empty, directed,} \\ \forall I \in \text{Idl}(\mathbb{A})[(\forall x' \in U \exists a' \in I, x' \leq \uparrow_{\mathbb{A}} a') \Rightarrow \exists a \in I, x \leq \uparrow_{\mathbb{A}} a]\}.$$

We now present several properties of dcpo presentations of the shape $\Delta(\mathbb{A})$. Let $\eta: \mathcal{F}(\mathbb{A}) \rightarrow \overline{\Delta(\mathbb{A})}$ be the natural map $x \mapsto \langle x \rangle$.

Lemma 10. *Let \mathbb{A} be a lattice. Then $\overline{\Delta(\mathbb{A})} = C_{\mathbb{A}}\text{-Idl}(\overline{\Delta(\mathbb{A})})$ and $\eta: \mathcal{F}(\mathbb{A}) \rightarrow \overline{\Delta(\mathbb{A})}$ is a \vee -homomorphism. Consequently, every $u \in \overline{\Delta(\mathbb{A})}$ is a lattice ideal of $\mathcal{F}(\mathbb{A})$.*

Proof. We will write Δ, \mathcal{F}, C , assuming \mathbb{A} is fixed.

We show the following stability property of C : for all $y \in \mathcal{F}$ and all $x \triangleleft U$, we have $x \vee y \triangleleft U \vee y$ where $U \vee y = \{x' \vee y \mid x' \in U\}$. To this end, suppose that $I \in \text{Idl}(\mathbb{A})$ such that for all $x' \in U$ there exists $a' \in I$ such that $x' \vee y \sqsubseteq \uparrow_{\mathbb{A}} a'$. Since U is non-empty, this condition is non-vacuous so that $y \sqsubseteq x' \vee y \sqsubseteq \uparrow_{\mathbb{A}} a'$ for some $x' \in U$ and $a' \in I$. Moreover, since $x \triangleleft U$ and $x' \sqsubseteq x' \vee y$ for all $x' \in U$, there exists $a \in I$ such that $x \sqsubseteq \uparrow_{\mathbb{A}} a$. But then also $x \vee y \sqsubseteq \uparrow_{\mathbb{A}} a \vee \uparrow_{\mathbb{A}} a' = \uparrow_{\mathbb{A}}(a \vee a')$ where $a \vee a' \in I$, so that $x \vee y \triangleleft U \vee y$. It now follows by [14, Proposition 6.2] that $\overline{\Delta}$ is the suplattice presented by $\overline{\Delta}$ and that $\eta: \mathcal{F} \rightarrow \overline{\Delta}$ is a \vee -homomorphism. It follows by Proposition 1 that $\overline{\Delta} = C\text{-Idl}(\overline{\Delta})$.

Let $u \in \overline{\Delta}$; we will show that u is a lattice ideal of \mathcal{F} . It follows from Definition 1 that u is a down-set. Moreover, if $x, y \in u$, then $\eta(x), \eta(y) \subseteq u$, so that $\eta(x) \vee \eta(y) \subseteq u$. Since η is a \vee -homomorphism, $\eta(x \vee y) \subseteq u$, whence $x \vee y \in u$. It follows that u is a lattice ideal. \square

Remark 1. We would like to highlight that Lemma 10 above is a crucial step in allowing the lifting of operators. The canonical extension of a lattice is not just a dcpo completion but a suplattice completion of the free dual dcpo completion of the lattice. However, there is no equivalent of Proposition 4 for suplattice algebras (see [14, Sec. 4]). The lemma tells us that $\overline{\Delta(\mathbb{A})}$ is in fact also the suplattice presented by $\Delta(\mathbb{A})$ as its elements are *all* $C_{\mathbb{A}}$ -ideals of $\Delta(\mathbb{A})$. The description of this suplattice completion as a dcpo completion is crucial as it implies that Proposition 4 applies. Thus Lemma 10 tells us that we *can* lift inequations to suplattices with presentations of the shape $\Delta(\mathbb{A})$ since they are also dcpo presentations.

The following Lemma will allow us to show that $\overline{\Delta(\mathbb{A})}$ is in fact the canonical extension of \mathbb{A} .

Lemma 11. *Let $\eta: \mathcal{F}(\mathbb{A}) \rightarrow \overline{\Delta(\mathbb{A})}$ be the natural map $x \mapsto \langle x \rangle$.*

1. *For all $x \in \mathcal{F}(\mathbb{A})$, $\eta(x) = \downarrow_{\mathcal{F}(\mathbb{A})} x$, hence $\eta: \mathcal{F}(\mathbb{A}) \rightarrow \overline{\Delta(\mathbb{A})}$ is an embedding.*
2. *$\overline{\Delta(\mathbb{A})}$ is a complete lattice.*
3. *$\eta: \mathcal{F}(\mathbb{A}) \rightarrow \overline{\Delta(\mathbb{A})}$ is a \vee, \wedge -homomorphism.*
4. *For all directed $T \subseteq A$, $\bigvee_{b \in T} \langle \uparrow_{\mathbb{A}} b \rangle = \bigcup_{b \in T} \langle \uparrow_{\mathbb{A}} b \rangle$.*

Proof. We will write Δ, \mathcal{F}, C , assuming \mathbb{A} is fixed.

(1) We will show that $\downarrow_{\mathcal{F}(\mathbb{A})} x$ is a C -ideal, which is sufficient since necessarily $\downarrow_{\mathcal{F}(\mathbb{A})} x \subseteq \langle x \rangle$. Suppose that $y \triangleleft U$ and $U \subseteq \downarrow_{\mathcal{F}(\mathbb{A})} x$. If $a \in A$ such that $x \leq \uparrow_{\mathbb{A}} a$ then $\downarrow_{\mathbb{A}} a$ is an ideal of \mathbb{A} and for each $x' \in U$, $x' \leq x \leq \uparrow_{\mathbb{A}} a$, so by the definition of C , there is $a' \in \downarrow_{\mathbb{A}} a$ with $y \leq \uparrow_{\mathbb{A}} a'$. That is, $x \leq \uparrow_{\mathbb{A}} a$ implies $y \leq \uparrow_{\mathbb{A}} a$ and thus

$$y \leq \bigwedge \{ \uparrow_{\mathbb{A}} a \mid x \leq \uparrow_{\mathbb{A}} a \} = x$$

and $\downarrow_{\mathcal{F}(\mathbb{A})} x$ is a C -ideal.

(2) It follows from Lemma 10 that $\overline{\Delta}$ is complete lattice.

(3) It follows from Lemma 10 that η is a \vee -homomorphism. Let $S \subseteq \mathcal{F}$; we will show that $\bigwedge_{x \in S} \langle x \rangle = \langle \bigwedge S \rangle$. This follows immediately from the fact that $C\text{-Idl}(\Delta)$ is a closure system and (1) above:

$$\bigwedge_{x \in S} \langle x \rangle = \bigcap_{x \in S} \langle x \rangle = \bigcap_{x \in S} \downarrow_{\mathcal{F}(\mathbb{A})} x = \downarrow_{\mathcal{F}(\mathbb{A})} (\bigwedge S) = \langle \bigwedge S \rangle.$$

(4) Since $\bigcup_{b \in T} \langle \uparrow_{\mathbb{A}} b \rangle \subseteq \langle \bigcup_{b \in T} \langle \uparrow_{\mathbb{A}} b \rangle \rangle = \bigvee_{b \in T} \langle \uparrow_{\mathbb{A}} b \rangle$, it suffices to show that $\bigcup_{b \in T} \langle \uparrow_{\mathbb{A}} b \rangle$ is a C -ideal. Let $I := \downarrow_{\mathbb{A}} T$. Now suppose that $x \triangleleft U$ and $U \subseteq \bigcup_{b \in T} \langle \uparrow_{\mathbb{A}} b \rangle = \bigcup_{b \in T} \downarrow_{\mathcal{F}(\mathbb{A})} \langle \uparrow_{\mathbb{A}} b \rangle$; then for each $x' \in U$, there is a $b' \in I$ such that $x' \leq \uparrow_{\mathbb{A}} b'$. Since $x \triangleleft U$, it follows that there is some $b \in I$ such that $x \leq \uparrow_{\mathbb{A}} b$; since $I = \downarrow_{\mathbb{A}} T$, we may assume that $b \in T$. But then $x \in \bigcup_{b \in T} \langle \uparrow_{\mathbb{A}} b \rangle$; it follows that $\bigcup_{b \in T} \langle \uparrow_{\mathbb{A}} b \rangle$ is a C -ideal. \square

Remark 2. Analogous to the \wedge, \vee -homomorphism $\eta: \mathcal{F}(\mathbb{A}) \rightarrow \overline{\Delta(\mathbb{A})}$ we could also define a \bigvee, \bigwedge -homomorphism $\mu: \mathcal{I}(\mathbb{A}) \rightarrow \overline{\Delta(\mathbb{A})}$, where $\mathcal{I}(\mathbb{A})$ is the directed join-completion (or the ideal completion) of \mathbb{A} . We would then use the map $\mu: y \mapsto \bigvee_{b \in y} \langle b \rangle$.

Let $e: \mathbb{A} \rightarrow \overline{\Delta(\mathbb{A})}$ be the restriction of $\eta: \mathcal{F}(\mathbb{A}) \rightarrow \overline{\Delta(\mathbb{A})}$ to A , i.e.

$$e: a \mapsto \langle \uparrow_{\mathbb{A}} a \rangle = \downarrow_{\mathcal{F}(\mathbb{A})}(\uparrow_{\mathbb{A}} a).$$

Theorem 12. *Let \mathbb{A} be a lattice. Then the embedding $e: \mathbb{A} \rightarrow \overline{\Delta(\mathbb{A})}$ is the canonical extension of \mathbb{A} .*

Proof. We will write Δ, \mathcal{F}, C as before. First, observe that it follows from Proposition 7 and Lemma 11.1 that $e: \mathbb{A} \rightarrow \overline{\Delta}$ is an embedding.

Next, in order to prove that the embedding is dense, assume that $u, v \in \overline{\Delta}$ such that $u \not\subseteq v$. We will show that there are a filter F and an ideal I of \mathbb{A} such that $\bigwedge e[F] \subseteq u$, $\bigwedge e[F] \not\subseteq v$, $u \not\subseteq \bigvee e[I]$ and $v \subseteq \bigvee e[I]$. It follows from $u \not\subseteq v$ that there is some $x \in u \setminus v$, so that $\langle x \rangle \subseteq u$ and $\langle x \rangle \not\subseteq v$. Take $F := \{a \in A \mid x \leq \uparrow_{\mathbb{A}} a\}$, then $\langle x \rangle = \bigwedge e[F]$ and we have our first witness; we will use this same element $x \in u \setminus v$ to find a suitable ideal I . Now observe that v is a directed subset of \mathcal{F} by Lemma 10. If it were the case that $x \triangleleft v$, then since v is a C -ideal and $v \subseteq v$, it would follow that $x \in v$, contrary to our assumption. So it must be the case that $x \not\triangleleft v$ and thus, by the definition of the covering relation, there must be some ideal $I \subseteq A$ such that

$$\forall x' \in v, \exists a' \in I \text{ such that } x' \leq \uparrow_{\mathbb{A}} a', \text{ but } \forall a \in I, x \not\leq \uparrow_{\mathbb{A}} a. \quad (1)$$

We claim that $u \not\subseteq \bigvee e[I]$ and $v \subseteq \bigvee e[I]$. If the former were the case, then we would find that

$$x \in u \subseteq \bigvee e[I] = \bigvee_{a \in I} \langle \uparrow_{\mathbb{A}} a \rangle = \bigcup_{a \in I} \langle \uparrow_{\mathbb{A}} a \rangle = \bigcup_{a \in I} \downarrow_{\mathcal{F}(\mathbb{A})}(\uparrow_{\mathbb{A}} a),$$

where the last two equalities follow from Lemma 11. It now follows that $x \leq \uparrow_{\mathbb{A}} a$ for some $a \in I$, contradicting (1). Finally, given $x' \in v$ and $a' \in I$ such that $x' \leq \uparrow_{\mathbb{A}} a'$, we find that $\langle x' \rangle \subseteq \langle \uparrow_{\mathbb{A}} a' \rangle$, so that it follows from (1) that

$$v = \bigvee \{\langle x' \rangle \mid x' \in v\} \subseteq \bigvee \{\langle \uparrow_{\mathbb{A}} a' \rangle \mid a' \in I\} = \bigvee e[I].$$

Finally, for the compactness property, suppose that F and I are an arbitrary filter and ideal of \mathbb{A} such that $\bigwedge e[F] \subseteq \bigvee e[I]$; we must show that there exists $a \in I$ and $b \in F$ such that $b \leq a$. By Lemma 11.3, $\bigwedge e[F] = \langle \bigwedge F \rangle$, so we find that

$$\bigwedge F \in \langle \bigwedge F \rangle = \bigwedge e[F] \subseteq \bigvee e[I] = \bigcup_{a \in I} \downarrow_{\mathcal{F}(\mathbb{A})}(\uparrow_{\mathbb{A}} a),$$

where the second equality follows from Lemma 11.4 as before. It follows that $\bigwedge F \in \downarrow_{\mathcal{F}(\mathbb{A})}(\uparrow_{\mathbb{A}} a)$ for some $a \in I$, so by Definition 5.4, there is some $b \in F$ such that $b \leq a$. \square

Recall that if \mathbb{A} is a lattice and $e: A \rightarrow \mathbb{A}^\sigma$ is its canonical extension, the *closed elements* of \mathbb{A}^σ are defined as

$$K(\mathbb{A}^\sigma) := \{\bigwedge e[F] \mid F \subseteq A, F \text{ a filter}\}.$$

If we view $\overline{\Delta(\mathbb{A})}$ as the canonical extension of \mathbb{A} , then the closed elements correspond to the elements of $\mathcal{F}(\mathbb{A})$:

$$K(\overline{\Delta(\mathbb{A})}) = \{\langle x \rangle \mid x \in \mathcal{F}(\mathbb{A})\}.$$

This follows from the fact that for each $x \in \mathcal{F}(\mathbb{A})$, $\{a \in A \mid x \leq \uparrow_{\mathbb{A}} a\}$ is a filter and we have $x = \bigwedge \{\uparrow_{\mathbb{A}} a \mid x \leq \uparrow_{\mathbb{A}} a\}$, and the fact that $\eta: \mathcal{F}(\mathbb{A}) \rightarrow \overline{\Delta(\mathbb{A})}$ preserves all meets by Lemma 11.3.

Lemma 13. *Let $\mathbb{A}_1, \dots, \mathbb{A}_n, \mathbb{B}$ be lattices and let $f: \mathbb{A}_1 \times \dots \times \mathbb{A}_n \rightarrow \mathbb{B}$ be an operator. Then $f^{\mathcal{F}}: \mathcal{F}(\mathbb{A}_1) \times \dots \times \mathcal{F}(\mathbb{A}_n) \rightarrow \mathcal{F}(\mathbb{B})$ is cover-stable.*

Proof. We write $x_i \triangleleft_i U_i$ if $(x_i, U_i) \in C_{\mathbb{A}_i}$ and $x \triangleleft U$ if $(x, U) \in C_{\mathbb{B}}$. Let $1 \leq i \leq n$, $(x_1, \dots, x_n) \in \mathcal{F}(\mathbb{A}_1) \times \dots \times \mathcal{F}(\mathbb{A}_n)$ and $U_i \subseteq \mathcal{F}(\mathbb{A}_i)$ such that $x_i \triangleleft_i U_i$. We need to show that

$$f^{\mathcal{F}}(x_1, \dots, x_n) \triangleleft \{f^{\mathcal{F}}(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \mid y \in U_i\}. \quad (2)$$

We will write $f^{\mathcal{F}}(-, y, -)$ for an element of the right hand side set above. Let $I \in \text{Idl}(\mathbb{B})$ such that for every $y \in U_i$, there is some $a_y \in I$ such that $f^{\mathcal{F}}(-, y, -) \leq \uparrow_{\mathbb{B}} a_y$. We need to find some $c \in I$ such that $f^{\mathcal{F}}(-, x_i, -) \leq \uparrow_{\mathbb{B}} c$. Now since $f^{\mathcal{F}}$ is co-Scott continuous, it is also co-Scott continuous in its i th coordinate [1, Lemma 3.2.6]. Thus if we take $y \in U_i$ and write $y = \bigwedge \{\uparrow_{\mathbb{A}_i} b \mid y \leq \uparrow_{\mathbb{A}_i} b\}$, then

$$\begin{aligned} f^{\mathcal{F}}(-, y, -) &= f^{\mathcal{F}}(-, \bigwedge \{\uparrow_{\mathbb{A}_i} b \mid y \leq \uparrow_{\mathbb{A}_i} b\}, -) \\ &= \bigwedge_{b \in A_i, y \leq \uparrow_{\mathbb{A}_i} b} f^{\mathcal{F}}(-, \uparrow_{\mathbb{A}_i} b, -) \leq \uparrow_{\mathbb{B}} a_y. \end{aligned}$$

It follows by Definition 5.4 that there is some $b_y \in A_i$ such that $y \leq \uparrow_{\mathbb{A}_i} b_y$ and $f^{\mathcal{F}}(-, y, -) \leq f^{\mathcal{F}}(-, \uparrow_{\mathbb{A}_i} b_y, -) \leq \uparrow_{\mathbb{B}} a_y$. Let $I' \in \text{Idl}(\mathbb{A}_i)$ be the ideal generated by $\{b_y \mid y \in U_i\}$. Since $y \leq \uparrow_{\mathbb{A}_i} b_y \in I'$ for each $y \in U_i$ and $x_i \triangleleft_i U_i$, it follows that there is some $b \in I'$ such that $x_i \leq \uparrow_{\mathbb{A}_i} b$. By definition of I' , there exist $y_1, \dots, y_k \in U$ such that $x_i \leq \uparrow_{\mathbb{A}_i} b \leq \uparrow_{\mathbb{A}_i} b_{y_1} \vee \dots \vee \uparrow_{\mathbb{A}_i} b_{y_k}$. But then

$$\begin{aligned} f^{\mathcal{F}}(-, x_i, -) &\leq f^{\mathcal{F}}(-, \uparrow_{\mathbb{A}_i} b, -) \\ &\leq f^{\mathcal{F}}(-, \uparrow_{\mathbb{A}_i} b_{y_1} \vee \dots \vee \uparrow_{\mathbb{A}_i} b_{y_k}, -) \\ &= f^{\mathcal{F}}(-, \uparrow_{\mathbb{A}_i} b_{y_1}, -) \vee \dots \vee f^{\mathcal{F}}(-, \uparrow_{\mathbb{A}_i} b_{y_k}, -) \\ &\leq \uparrow_{\mathbb{B}} a_{y_1} \vee \dots \vee \uparrow_{\mathbb{B}} a_{y_k} = \uparrow_{\mathbb{B}} (a_{y_1} \vee \dots \vee a_{y_k}), \end{aligned}$$

where the first equality follows from the fact that $f^{\mathcal{F}}$ is an operator (by Proposition 8). Since $a_{y_1} \vee \dots \vee a_{y_k} \in I$ and I was arbitrary, it follows that (2) holds. \square

Corollary 14. *Let $\mathbb{A}_1, \dots, \mathbb{A}_n$ and \mathbb{B} be lattices and let $f: \mathbb{A}_1 \times \dots \times \mathbb{A}_n \rightarrow \mathbb{B}$ be an operator. Then $f^{\mathcal{F}}: \overline{\Delta(\mathbb{A}_1)} \times \dots \times \overline{\Delta(\mathbb{A}_n)} \rightarrow \overline{\Delta(\mathbb{B})}$ is well-defined and Scott-continuous. Moreover, $f^{\mathcal{F}} = f^{\sigma}$.*

Proof. Let $f: \mathbb{A}_1 \times \cdots \times \mathbb{A}_n \rightarrow \mathbb{B}$ be as in the assumptions above. It follows from Proposition 8 and Lemma 13 that $\overline{f^{\mathcal{F}}}$ is well-defined and Scott-continuous. To show that $\overline{f^{\mathcal{F}}} = f^\sigma$; observe that $\overline{f^{\mathcal{F}}}$ and f^σ agree on closed elements:

$$\overline{f^{\mathcal{F}}}(\langle x_1 \rangle, \dots, \langle x_n \rangle) = \langle f^{\mathcal{F}}(x_1, \dots, x_n) \rangle,$$

by [14, Lemma 3.3]. Since $x_i = \bigwedge \{\uparrow_{\mathbb{A}_i} b \mid x_i \leq \uparrow_{\mathbb{A}_i} b \text{ for all } 1 \leq i \leq n\}$, we find that

$$\begin{aligned} \langle f^{\mathcal{F}}(x_1, \dots, x_n) \rangle &= \\ \langle f^{\mathcal{F}}(\bigwedge \{\uparrow_{\mathbb{A}_1} a_1 \mid x_1 \leq \uparrow_{\mathbb{A}_1} a_1\}, \dots, \bigwedge \{\uparrow_{\mathbb{A}_n} a_n \mid x_n \leq \uparrow_{\mathbb{A}_n} a_n\}) \rangle &= \\ \bigwedge \{\langle \uparrow_{\mathbb{B}} f(a_1, \dots, a_n) \rangle \mid (x_1, \dots, x_n) \sqsubseteq (\uparrow_{\mathbb{A}_1} a_1, \dots, \uparrow_{\mathbb{A}_n} a_n)\} &= f^\sigma(\langle x_1 \rangle, \dots, \langle x_n \rangle), \end{aligned}$$

where the second equality follows from the fact that both $f^{\mathcal{F}}$ and $\langle \cdot \rangle$ commute with co-directed meets.

Secondly, recall from Lemma 10 that every $u \in \overline{\Delta(\mathbb{A}_i)}$, seen as a C -ideal, is a directed subset of $\mathcal{F}(\mathbb{A})$. Thus, $u = \bigvee_{x \in u} \langle x \rangle$ is a directed join. Since we showed above that $\overline{f^{\mathcal{F}}}$ is Scott-continuous, it follows that

$$\begin{aligned} \overline{f^{\mathcal{F}}}(u_1, \dots, u_n) &= \overline{f^{\mathcal{F}}}(\bigvee_{x_1 \in u_1} \langle x_1 \rangle, \dots, \bigvee_{x_n \in u_n} \langle x_n \rangle) = \\ \bigvee \{\overline{f^{\mathcal{F}}}(\langle x_1 \rangle, \dots, \langle x_n \rangle) \mid x_i \in u_i \text{ for all } 1 \leq i \leq n\} &= \\ \bigvee \{f^\sigma(\langle x_1 \rangle, \dots, \langle x_n \rangle) \mid x_i \in u_i \text{ for all } 1 \leq i \leq n\} &= f^\sigma(u_1, \dots, u_n), \end{aligned}$$

for arbitrary $(u_1, \dots, u_n) \in \overline{\Delta(\mathbb{A}_1)} \times \cdots \times \overline{\Delta(\mathbb{A}_n)}$. \square

Thus, we have shown that the dcpo presentation $\Delta(\mathbb{A})$ of Definition 8 allows us to describe the canonical extension of a lattice \mathbb{A} , together with the σ -extension of any additional operator $f: A^n \rightarrow A$. The following theorem, which can be found in [7, 6], can now be seen as an application of general results concerning dcpo algebras from [14] to the specific case of canonical extensions of lattices with operators.

Theorem 15 (cf. [7], Theorem 4.5 and [6], Theorem 6.3). *Let $\mathbb{A} = \langle A; \wedge_{\mathbb{A}}, \vee_{\mathbb{A}}, 0_{\mathbb{A}}, 1_{\mathbb{A}}, (\omega_{\mathbb{A}})_{\omega \in \Omega'} \rangle$ be a bounded lattice with additional operations and let $\Omega \subseteq \{\wedge, \vee, 0, 1\} \cup \Omega'$ consist entirely of operation symbols that interpret as operators in \mathbb{A} . If $s(x_1, \dots, x_n)$ and $t(x_1, \dots, x_n)$ are n -ary Ω -terms such that $\mathbb{A} \models s \preceq t$, then also $\mathbb{A}^\sigma \models s \preceq t$.*

Proof. Let \mathbb{A} , s and t be as in the assumptions of the theorem. Since operators are monotone, it follows by Proposition 6 that $\mathcal{F}(\mathbb{A}) \models s \preceq t$. It follows by Proposition 8 and Lemma 13 that $\overline{\Delta(\mathbb{A})} \models s \preceq t$. \square

Remark 3. Observe that $\vee_{\mathbb{A}}: \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$ is always an operator by associativity but that $\wedge_{\mathbb{A}}: \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$ is an operator if and only if \mathbb{A} is distributive.

Remark 4. Canonical extension is a two sided construction: it does not favour joins over meets. This is perhaps best illustrated by [10]. There it is shown that if we consider alternating applications of directed join and meet completion to a lattice \mathbb{A} , then the embeddings $\downarrow_{\mathcal{F}(\mathbb{A})}: \mathcal{F}(\mathbb{A}) \rightarrow \mathcal{I}(\mathcal{F}(\mathbb{A}))$ and $\uparrow_{\mathcal{I}(\mathbb{A})}: \mathcal{I}(\mathbb{A}) \rightarrow \mathcal{F}(\mathcal{I}(\mathbb{A}))$ factor through \mathbb{A}^δ in a unique way; see Figure 1. In order to apply

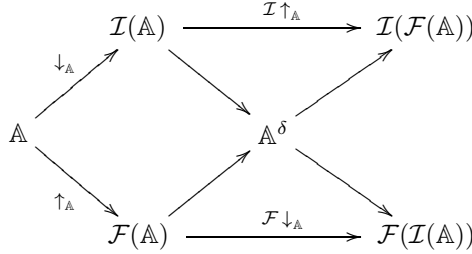


Figure 1: The canonical extension as an interpolant, as discussed in [10]

the existing theory on dcpo completions we have presented our results in terms of a dcpo completion of the free co-directed meet completion of the original lattice, using the fact that \mathbb{A}^δ interpolates between $\mathcal{F}(\mathbb{A})$ and $\mathcal{I}(\mathcal{F}(\mathbb{A}))$. Of course the order dual approach would have worked just as well: Starting from the directed join completion (concretely, the ideal completion) of \mathbb{A} , we could have given a *co-dcpo*-presentation of \mathbb{A}^δ . The extension of a dual operator $f: \mathbb{A}_1 \times \dots \times \mathbb{A}_n \rightarrow \mathbb{B}$, i.e. a map preserving binary meets in each coordinate, via this co-dcpo presentation would then yield an extension $f^\pi: \mathbb{A}_1^\delta \times \dots \times \mathbb{A}_n^\delta \rightarrow \mathbb{B}$ of f and the dual of Theorem 15 would guarantee that equations among dual operators lift to the extension. This remark restores some symmetry to the situation, though we note that the extension f^σ obtained from the free co-dcpo followed by the dcpo completion described in this paper and the extension of an operation obtained via the order dual approach do not in general agree. This latter extension is also well known and much used in the theory of canonical extensions and is known as the π -extension of f . The extension of the underlying lattice using either approach is however one and the same – this is easy to see by the fact that the characterising properties of canonical extensions are self-dual properties.

4. Discussion

The original 1951 canonicity result of Jónsson and Tarski had a fairly complicated proof. In addition, it required the underlying lattice to be, not only distributive, but Boolean even though the canonicity of equations only is implied if the negation is not involved. The latter fact obviously begged the question of whether the result was actually a (distributive) lattice result.

It took over 40 years before this question was answered in the positive in the paper [7] (and fairly soon afterwards, it was shown [6] that it was in fact just a

lattice result). The main breakthrough was in the 1994 paper and it consisted in realising the central role played by Scott continuity. Even though the paper [7] was written in a language quite different from that of [14], the general lines of the proof in [7] do in fact follow those of [14], albeit in the special case of the presentation $\Delta(\mathbb{A})$. With this article we have shown explicitly how the two relate.

While the canonicity result for operators is a special case of the much more general domain theoretic result of [14], the real power and interest of canonical extensions involves, at least the presence, and sometimes also the direct involvement of *order reversing* operations such as negations, implications, and other non-monotonic logical connectives. Because of the up-down symmetry of canonical extension, order-reversing operations are easily and meaningfully extended to canonical extensions (we have just identified it as the free dcpo generated by a dcpo presentation based on a free co-dcpo completion, but as mentioned in Remark 4 above, we could as well have obtained it as the free co-dcpo generated by a co-dcpo presentation based on a free dcpo completion of the original algebra). In [8] topological methods for canonical extensions were introduced and these allow arbitrary maps to be extended to the canonical extension in a very natural way. This in turn allows for a very fine analysis of canonicity in that general setting [8]. We are not aware of any parallel to these methods in domain theory but expect that the current paper will foster new unifying developments.

As a case in point, one of the referees of this paper pointed out that our Definition 8, and the results following it, may be generalised to a more general dcpo presentation setting. These generalisations are indeed possible and this is closely related to parallel work of Sam van Gool on canonical extensions of strong proximity lattices which are a kind of dcpo presentations of stably compact spaces.

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